Generalized solutions of the Cauchy problem for the Navier-Stokes system and diffusion processes

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Abstract

We reduce the construction of a weak solution of the Cauchy problem for the Navier-Stokes system on R^3 to the construction of a solution to a stochastic problem. Namely, we construct diffusion processes which allow us to obtain a probabilistic representation of a weak (in distributional sense) solution to the Cauchy problem for the Navier-Stokes system on a small time interval. Strong solutions on a small time interval are constructed as well

Introduction

The main purpose of this article is to construct both strong and weak solutions (in certain functional classes) of the Cauchy problem for the Navier-Stokes (N-S) system in R^3. To this end we consider a stochastic problem and show that the solution of the Cauchy problem for the Navier-Stokes system can be constructed via the solution of this stochastic problem.

The approach we develop in this article is based on the theory of stochastic equations associated with nonlinear parabolic equations started by McKean [1] and Freidlin [2],[3] and generalized by Belopolskaya and Dalecky [4], [5] on one hand and on the theory of stochastic

1

flows due to Kunita [6] on the other hand. In our previous paper [7] we have constructed a stochastic process that allows us to prove the existence and uniqueness of a local in time classical (C^2-smooth in the spatial variable) solution of the Cauchy problem for the Navier -Stokes system. In the present paper we construct a process which allows us to obtain construction of solutions of the both weak and strong Cauchy problem for this system. Later we plan to apply a similar approach for the Navier-Stokes equation for compressible fluids extending the results from [9], [10].

A close but different approach is the Euler-Lagrange approach to incompressible fluids which was developed by Constantin [11] and Constantin and Iyer [12]. Shortly, the main differences in these approaches are the following: we use a probabilistic representation for the Euler pressure instead of the Leray projection and obtain different formulas for the stochastic representation of the velocity field. We discuss these differences with more details in the last section of the present work.

Thestructure of the present article is as follows. In the first section we give some preliminary information concerning different analytical approaches to the Navier-Stokes system. Here we recall some common ways to eliminate the pressure and to obtain a closed equation for the velocity.

The classical approaches here are based on the so called Leray (Leray-Hodge)-projection that is a projection of the space of square integrable vector fields to the space of divergence free square integrable vector fields. Applying such a projection to the velocity equation one can eliminate the pressure p and get the closed equation for the velocity u. This operator is used both in numerous analytical papers (see [15] for references) and in papers where the N-S system is studied from the probabilistic point of view [16],[17], [12]. Finally the pressure is reconstructed from the Poisson equation.

One more possibility to eliminate the pressure appears when one considers the equation for the vorticity of the velocity field u and uses the Biot-Sawart law to obtain a closed system. From the probabilistic point of view this approach was investigated in [18].

In our previous paper [7] we do not use the Leray projection but instead we start with consideration of a system consisting of the original velocity equation and the Poisson equation for the pressure and construct their probabilistic counterpart. The probabilistic counterpart of the N-S system was presented in the form of a system of stochastic equations. Furthermore we prove the existence and uniqueness of a solution to this stochastic system and show that in this way we construct a unique classical (strong) solution of the Cauchy problem for the N-S system defined on a small time interval depending on the Cauchy data.

In the present paper we also reduce the N-S system to the system of equations consisting of the original velocity equation and the Poisson equation for the pressure but then an associated stochastic problem considered here allows to construct a generalized (distributional) solution to the Cauchy problem for the N-S system. The as-

sociated stochastic problem is studied in section 5. In sections 1-4 we expose auxiliary results used in section 5. Namely, in section 1 we give analytical preliminaries and recall the notions of strong, weak and mild solutions to the Cauchy problem for the Navier-Stokes system. More detail can be found the recent book by Lemarie-Rieusset [15]. In section 2 we give a short review of probabilistic approaches to the investigation of the Navier-Stokes system [7], [16] -[18]. In section 3 we study a probabilistic representation of the solution to the Poisson equation, while in section 4 we recall some principal fact of the Kunita theory of stochastic flows and apply the results from [19], [20] to construct a solution of the Cauchy problem for a nonlinear parabolic equation (see also [21]). Finally all these preliminary results are used to construct the probabilistic counterpart of the Navier-Stokes system, prove that there exists a unique local solution to the corresponding stochastic system and apply the results to construct both the strong and weak (and simultaneously mild) solutions to the Cauchy problem for the Navier-Stokes system.

1 Preliminaries

As it was mentioned in the introduction the main purpose of this article is to construct both strong and weak solutions (in certain functional classes) of the Cauchy problem for the Navier-Stokes system via diffusion processes.

Consider the Cauchy problem for the Navier-Stokes system

$$\frac{\partial u}{\partial t} + (u, \nabla)u = \nu \Delta u - \nabla p, \quad u(0, x) = u_0(x), \quad x \in R^3, \qquad (1.1)$$

$$div\, u = 0. \qquad (1.2)$$

Here $u(t, x) \in R^3, x \in R^3, t \in [0, \infty)$ is the velocity of the fluid at the position x at time t and $\nu > 0$ is the viscosity coefficient and $p(t, x)$ is a scalar field called the pressure which appears in the equation to enforce the incompressibility condition (1.2). Later we set $\nu = \frac{\sigma^2}{2}$ for reasons to be explained below.

By eliminating the pressure from (1.1),(1.2) one gets a nonlinear pseudo-differential equation which is to be solved. There exist different ways to do it and we consider now some of them.

Given a vector field f let $\mathbf{P}f$ be given by

$$\mathbf{P}f = f - \nabla \Delta^{-1} \nabla \cdot f. \qquad (1.3)$$

Here and below we denote by $u \cdot v$ the inner product of vectors u and v valued in R^3.

The map $\mathbf{P}$ called the Leray projection is a projection of the space $\mathbf{L}^2(R^3) \equiv L^2(R^3)^3$ of square integrable vector fields to the space of divergence free vector fields and we discuss its properties below. A quite direct definition of $\mathbf{P}$ is connected with the Riesz transformation R_j. Recall that $R_k = \frac{\nabla_k}{\sqrt{-\Delta}}$ which means that for $f \in \mathbf{L}^2$ we have

$\mathcal{F}(R_j f) = \frac{i\xi_j}{|\xi|}\hat{f}(\xi)$ where $\mathcal{F}(f) = \hat{f}$ is the Fourier transform of f. Then $\mathbf{P}$ is defined on $\mathbf{L}^2(R^3)$ as $\mathbf{P} = Id + R \otimes R$ or

$$(\mathbf{P}f)_j = f_j + \sum_{k=1}^{3} R_j R_k f_k.$$

Since $R_k R_j$ is a Calderon-Zygmund operator, $\mathbf{P}f$ may be defined on many Banach spaces.

Set

$$\gamma(t,x) = \sum_{k,j=1}^{3} \nabla_k u_j \nabla_j u_k = \text{Tr}[\nabla u]^2 \qquad (1.4)$$

and note that γ can be presented as well in the form

$$\gamma = \nabla \cdot \nabla \cdot u \otimes u = \sum_{j,k} \nabla_k \nabla_j (u_k u_j).$$

.

By computing the divergence of both parts of (1.1) and taking into account (1.2) we derive the equation

$$-\Delta p(t,x) = \gamma(t,x) \qquad (1.5)$$

thus arriving at the Poisson equation. The formal solution of the Poisson equation is given by

$$p = \Delta^{-1}\gamma = \Delta^{-1}\nabla \cdot \nabla \cdot u \otimes u \qquad (1.6)$$

since $\text{div}\, u = 0$ and finally we present ∇p in the form

$$\nabla p = \nabla \Delta^{-1}\nabla \cdot \nabla \cdot u \otimes u.$$

Substituting this expression for ∇p into (1.1) we obtain the following Cauchy problem

$$\frac{\partial u}{\partial t} = \nu \Delta u - \mathbf{P}\nabla \cdot (u \otimes u), \quad u(0) = u_0. \qquad (1.7)$$

There are a number of ways to define a notion of a solution for the Cauchy problem (1.7). We will appeal mainly to the Leray weak solution [13] or to the Kato mild solution [14].

1.1 Leray and Kato approaches to the solutions of the Navier-Stokes equations

Let $\mathcal{D} = \mathcal{D}(R^3) = C_c^\infty$ be the space of all infinitely differentiable functions on R^3 with compact support equipped with the Schwartz topology. Let $\mathcal{D}'$ be the topological dual of $\mathcal{D}$ and denote by $\langle \phi, \psi \rangle = \int_{R^3} \phi(x)\psi(x)dx$ the natural coupling between $\phi \in \mathcal{D}$ and $\psi \in \mathcal{D}'$. If it will not lead to misunderstandings we will use the same notation for vector fields u and v as well, that is

$$\langle h, u \rangle = \int_{R^3} \sum_{k=1}^{3} h_k(x)u_k(x)dx.$$

We recall that a weak solution of the N-S system on $[0, T] \times R^3$ is a distribution vector field $u(t, x)$ in $(\mathcal{D}'((0, T) \times R^3))^3$ where u is locally square integrable on $(0, T) \times R^3$, div $u = 0$ and there exists $p \in \mathcal{D}'((0, T) \times R^3)$ such that

$$\frac{\partial u}{\partial t} = \nu \Delta u - \nabla \cdot (u \otimes u) - \nabla p, \quad \lim_{t \to 0} u(t) = u_0 \qquad (1.8)$$

holds.

The Leray solution to the N-S equations is constructed through a limiting procedure from the solutions to the mollified N-S equations

$$\begin{cases} \frac{\partial u}{\partial t} = \nu \Delta u - \nabla \cdot ((u * q_\varepsilon) \otimes u) - \nabla p, \\ \nabla \cdot u = 0, \\ \lim_{t \to 0} u(t) = u_0. \end{cases} \qquad (1.9)$$

Namely it is proved that there exists a function

$$u_\varepsilon \in L^\infty((0, \infty), \mathbf{L}^2) \cap L^2((0, T), (\dot{\mathbf{H}}^1))$$

such that (at least for a subsequence u_{ε_k}) strongly converging in $(L^2_{loc}((0, T) \times R^3))^3$ to u which satisfies (1.9).

Here $\dot{\mathbf{H}}^1$ is the homogenous Sobolev space $\dot{\mathbf{H}}^1 = \{f \in \mathbf{S}'_0 : \nabla f \in \mathbf{L}^2\}$ with norm $\|f\|_{\mathbf{H}_1} = \|\nabla f\|_{\mathbf{L}_2}$.

On the other hand to construct the Kato solution means to construct a solution u to the following integral equation

$$u(t) = e^{t\Delta} u_0 - \int_0^t e^{(t-s)\Delta} \mathbf{P} \nabla \cdot (u \otimes u)(s) ds. \qquad (1.10)$$

Note that instead of looking for $u(t, x)$ and $p(t, x)$ one can prefer to look for their Fourier images $\hat{u}(t, \lambda) = (2\pi)^{-\frac{3}{2}} \int_{R^3} e^{-i\lambda \cdot x} u(t, x) dx$.

The Leray and Kato approaches stated in terms of the Fourier transformations of the Navier-Stokes system can be described as follows.

Applying the Fourier transformation to the relation (1.7) written in the form

$$\langle h, u(t) \rangle = \langle h, u(0) \rangle + \int_0^t \langle h, \Delta u(s) \rangle - \int_0^t \langle h, \nabla \cdot (u \otimes u)(s) \rangle$$

we derive the relation

$$\langle \hat{h}, \hat{u} \rangle = \langle \hat{h}, \hat{u}_0 \rangle - \int_0^t \langle \hat{h}, |\lambda|^2 \hat{u}(s) \rangle ds - \qquad (1.11)$$

$$\frac{i}{(2\pi)^{\frac{3}{2}}} \int_0^t \int_{R^3} \int_{R^3} \sum_{k,l=1}^3 \lambda^k \hat{h}^l(\lambda'), \hat{u}_l(s, \lambda) u_k(s, \lambda - \lambda') d\lambda d\lambda' ds.$$

Here $\hat{u}$ corresponds to the Fourier transformation of u.

On the other hand if we are interested in the Kato mild solution of the N-S system then we may apply the Fourier transformation to (1.10) and derive the following equation

$$\chi(t, \lambda) = exp\{-\nu|\lambda|^2 t\}\chi(0, \lambda) + \qquad (1.12)$$

5

$$\int_0^t \nu |\lambda|^2 e^{-\nu |\lambda|^2 (t-s)} \left[\frac{1}{2} (\chi(s) \circ \chi(s)) \right] (\lambda) ds$$

for the function

$$\chi(t,\lambda) = \frac{2}{\nu} \left(\frac{\pi}{2} \right)^{\frac{3}{2}} |\lambda|^2 \hat{u}(t,\lambda).$$

Here

$$\chi_1 \circ \chi_2 (\lambda) = -\frac{i}{\pi^3} \int_{R^3} (\chi_1(\lambda_1) \cdot e_\lambda) \Pi(\lambda) \chi_2(\lambda - \lambda') \frac{|\lambda| d\lambda'}{|\lambda'|^2 |\lambda - \lambda'|^2}, \quad (1.13)$$

$e_\lambda = \frac{\lambda}{|\lambda|}$ and

$$\Pi(\lambda)\chi = \chi - e_\lambda(\chi \cdot e_\lambda), \quad (1.14)$$

Coming back to (1.7) we note that the Leray projection allows to eliminate the pressure $p(t,x)$ from the Navier-Stokes system, to construct u and finally to look for p defined by the solution of the auxiliary Poisson equation.

Another way to eliminate $p(t,x)$ from the system (1.1),(1.2) is to consider the function $v(t,x) = \operatorname{curl} u(t,x)$ called the vorticity. Since $\operatorname{curl} \nabla p(t,x) = 0$ one can derive a closed system for u and v. Namely for u and v we arrive at the system consisting of the equation

$$\frac{\partial v}{\partial t} + (u \cdot \nabla) v = \nu \Delta v + (v \cdot \nabla) u, \quad (1.15)$$

and the so called Biot-Savart law having the form

$$u(t,x) = \frac{1}{4\pi} \int_{R^3} \frac{(x-y) \times v(y)}{|x-y|^3} dy. \quad (1.16)$$

Here the cross-product $u \times v$ is given by

$$u \times v = \det \begin{pmatrix} e_1 & e_2 & e_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{pmatrix} =$$

$$(u_2 v_3 - u_3 v_2) e_1 + (u_3 v_1 - u_1 v_3) e_2 + (u_1 v_2 - u_2 v_1) e_3,$$

where (e_1, e_2, e_3) is the orthonormal basis in R^3.

Note that the term $(v \cdot \nabla) u$ can be written as $(\nabla u) v$ or even as $\mathcal{D}_u v$, where $\mathcal{D}_u$ is the deformation tensor defined as the symmetric part of ∇u

$$\mathcal{D}_u = \frac{1}{2} (\nabla u + \nabla u^T),$$

since by direct computation we see that

$$(\nabla u) v - \mathcal{D}_u v = \frac{1}{2} (\nabla u + \nabla u^T) v = 0.$$

To be able to present the precise statements concerning the existence and uniqueness of solutions to the N-S equations we have to introduce a number of functional spaces to be used in the sequel.

1.2 Functional spaces

We describe here functional spaces which will be used in the sequel.

Let $\mathcal{D} = \mathcal{D}(R^3)$ be the space of all infinitely differentiable functions on R^3 with compact supports equipped with the Schwartz topology. Let $\mathcal{D}'$ be the topological dual to $\mathcal{D}$. The elements of $\mathcal{D}'$ are called Schwartz distributions.

The space of R^3-valued vector fields h with components $h_k \in \mathcal{D}$ shall be denoted by $\mathbf{D}(R^3)$ and $\mathbf{D}'$ shall denote the space dual to $\mathbf{D}(R^3)$.

Let $L^q(R^3)$ denote the Banach space of functions f which are absolutely integrable taken to the q-th power with the norm $\|f\|_q = (\int_{R^3} |f(x)|^q dx)^{\frac{1}{q}}$;

Let Z denote the set of all integers, and suppose that $k \in Z$ is positive and $1 < q < \infty$. Denote by $W^{k,q} = W^{k,q}(R^3)$ the set of all real functions h defined on R^3 such that h and all its distributional derivatives ∇^α of order $|\alpha| = \sum \alpha_j \leq k$ belong to $L^q(R^3)$. It is a Banach space with norm

$$\|h\|_{k,p} = (\sum_{|\alpha| \leq k} \int_{R^3} |D^\alpha h(x)|^q dx)^{\frac{1}{q}}. \tag{1.17}$$

We denote the dual space of $W^{k,q}$ by $W^{-k,m}$ where $\frac{1}{m} + \frac{1}{q} = 1$. Elements of $W^{-k,q}$ can be identified with Schwartz distributions. The space $W^{-k,q}$ is also a Banach space with norm

$$\|\phi\|_{-k,q} = \sup_{\|h\|_{k,q} \leq 1} |\langle \phi, h \rangle|,$$

where

$$\langle \phi, h \rangle = \int_{R^3} \phi(x) h(x) dx.$$

The spaces $W^{k,p}$ for $k \in Z$ and $p > 1$ are called Sobolev spaces. If $p = 2$ we use the notation H^k for the Hilbert spaces $W^{k,2}$. In a natural way one can define the spaces $\mathbf{W^{k,q}}$, $\mathbf{H}^k$ of vector fields with components in $W^{k,p}$, and H^k and so on.

Set

$$\mathcal{V} = \{v \in \mathbf{D} : div v = 0\}$$

and let

$$\mathbf{H} = \{\text{closure of } \mathcal{V} \text{ in } \mathbf{L}^2(R^3)\}, \quad \mathbf{V} = \{\text{closure of } \mathcal{V} \text{ in } \mathbf{H}^1\}. \tag{1.18}$$

Let $C_b^k(R^3, R^3)$ denote the space of k-times differentiable fields with the norm

$$\|g\|_{C_b^k} = \sum_{|\beta| \leq k} \|D^\beta g\|_\infty$$

and let $C_b^{k,\alpha}(R^3, R^3)$ be the space of vector fields whose k-th derivatives are Hölder continuous with exponent α, $0 < \alpha < 1$ with the norm

$$\|g\|_{C_b^{k,\alpha}} = \|g\|_{C_b^k} + [g]_{k+\alpha}$$

where
$$[g]_{k+\alpha} = \sum_{|\beta|=k} \sup_{x,y\in R^3} \frac{|D^\beta g(x) - D^\beta g(y)|}{|x-y|^\alpha}.$$

We denote by $\text{Lip}(R^3)$ the space of bounded Lipschitz continuous functions with the norm
$$\|g\|_{Lip} = \sup_{x,y\in R^3} \frac{|g(x) - g(y)|}{|x-y|}.$$

Spaces of integrable functions on the whole R^3 appear to be not satisfactory to construct a solution to the N-S equations and one has to consider spaces of locally integrable functions.

Let $f : R^3 \to R^1$ be a Lebesgue measurable function. A set of functions $\{f : \int_K |f(x)|^p dx < \infty\}$ for all compact subsets K in R^3 is denoted by L^p_{loc} and called a space of locally integrable functions. Note that $L^1(R^3) \subset L^1_{loc}(R^3)$. Although $L^p_{loc}(R^3)$ are not normed spaces they are readily topologized. Namely a sequence $\{u_n\}$ converges to u in $L^p_{loc}(R^3)$ if $\{u_n\} \to u$ in $L^p(K)$ for each open $K \subset G$ having compact closure in R^3. Local spaces $W^{k,p}_{loc}(R^3)$ can be defined to consist of functions belonging to $W^{k,p}(K)$ for all compact $K \subset R^3$.

A local space $W^{k,p}_{loc}(G)$ is defined as a space of functions belonging to $W^{k,p}(G')$ for all $G' \subset G$ with compact closure in G. A function $f \in W^{k,p}_{loc}(G)$ with compact support will in fact belong to $W^{k,p}_0(G)$. Also functions in $W^{1,p}(G)$ which vanish continuously on the boundary ∂G will belong to $W^{1,p}_0(G)$ since they can be approximated by functions with compact support.

In the whole space R^3 and with p, q satisfying $1 \leq q \leq p < \infty$ denote by $\mathcal{M}^p_q$ a nonhomogenous Morrey space and by M^p_q a homogenous Morrey space with norms given respectively by

$$\mathcal{M}^p_q = \left\{ f \in L^q_{loc} : \|f\|_{\mathcal{M}^p_q} = \sup_{x_0\in R^3} \sup_{0<R} R^{\frac{3}{p}-\frac{3}{q}} \|f\|_{L^q(B(x_0,R))} < \infty \right\},$$
(1.19)

and

$$M^p_q = \left\{ f \in L^q_{loc} : \|f\|_{\mathcal{M}^p_q} = \sup_{x_0\in R^3} \sup_{0<R\leq 1} R^{\frac{3}{p}-\frac{3}{q}} \|f\|_{L^q(B(x_0,R))} < \infty \right\}$$
(1.20)

where $B(x_0, R)$ is a closed ball of R^3 with center at x_0 and radius R.

Respectively the integrable function is said to belong to $M^q(G)$ if there exists a constant C such that
$$\int_{G\cap B_R} |f(x)|dx \leq CR^{3(1-\frac{1}{q})}$$
(1.21)

for all balls B_R. The norm in $M^q(G)$ is defined as the minimum of the constants C satisfying (1.21)

A distribution u on $(0,T)\times R^3$ is said to be uniformly locally square integrable if for all $\varphi \in \mathcal{D}((0,T) \times R^3)$

$$\sup_{x_0\in R^3} \int_0^T \int_{R^3} \||\varphi(t, x-x_0)u(t,x)\||^2 dxdt < \infty.$$

8

Equivalently u is uniformly locally square integrable if and only if for all $t_0 < t_1 \in (0, T)$ the function $U_{t_0, t_1}(x) = (\int_{t_0}^{t_1} \|u(t, x)\|^2 dt)^{\frac{1}{2}}$ belongs to the Morrey space L^2_{uloc} . In this case we write

$$u \in \cap_{0 < t_0 < t_1 < T}(\mathbf{L}^2_{uloc}\mathbf{L}^2_t((t_0, t_1) \times R^3)).$$

For $1 \le p \le \infty$ the Morrey space of uniformly locally integrable functions on R^3 is the Banach space L^p_{uloc} of Lebesgue measurable functions f on R^3 such that the norm $\|f\|_{p,uloc}$ is finite, where

$$\|f\|_{p,uloc} = \sup_{x_0 \in R^3} (\int_{\|x - x_0\| < 1} |f(x)|^p dx)^{\frac{1}{p}}.$$

For $t_0 < t_1$, $1 \le p, q \le \infty$ the space $L^p_{uloc,x}L^q_t((t_0, t_1) \times R^3))$ is the Banach space of Lebesgue measurable functions f on $(t_0, t_1) \times R^3$ such that the norm

$$\sup_{x_0 \in R^3} (\int_{\|x - x_0\| < 1} (\int_{t_0}^{t_1} |f(t, x)|^q dt)^{\frac{p}{q}} dx)^{\frac{1}{p}}$$

is finite.

A C^∞ function f on R^3 is called rapidly decreasing if

$$\lim_{x \to \infty} |D^\alpha f(x)|(1 + \|x\|)^n = 0$$

holds for any multi-index α and any positive integer n. Let $\mathcal{S} = \mathcal{S}(R^3)$ be the space of rapidly decreasing $C^\infty-$ functions equipped with the Schwartz topology and $\mathcal{S}'$ be the topological dual of $\mathcal{S}$. Since $\mathcal{S}$ includes $\mathcal{D}$, $\mathcal{S}'$ is a subset of $\mathcal{D}'$. The elements of $\mathcal{S}'$ are called tempered distributions.

1.3 Weak, strong and mild solutions of the Navier-Stokes system

Now we are ready to give more precise definitions and statements concerning the existence and uniqueness of solutions of the N-S equations.

Definition 1.1.(Weak solutions) *A weak solution of the Navier-Stokes system on $(0, T) \times R^3$ is a distribution vector field $u(t, x)$, $u \in (\mathcal{D}'((0, T) \times R^3)^d$ such that*

a) u is locally square integrable on $(0, T) \times R^3$,
b) $\nabla \cdot u = 0$,
c) there exists $p \in \mathcal{D}'((0, T) \times R^3)$ such that

$$\partial_t u = \Delta u - \nabla \cdot (u \otimes u) - \nabla p.$$

The classical results concerning the existence of square integrable weak solutions are due to Leray [13].

Theorem 1.1. (Leray's theorem) *Let $u_0 \in (L^2(R^3))^3$ so that $\nabla \cdot u = 0$. Then there exists a weak solution $u \in L^\infty((0, \infty), (L^2)^d) \cap L^2((0, \infty), (H^1)^3)$ for the Navier -Stokes equation on $(0, \infty) \times R^3$ so*

that $lim_{t\to 0}\|u(t) - u_0\|_2 = 0$. Moreover, the solution u satisfies the energy inequality

$$\|u(t)\|^2 + 2\int_0^t \int_{R^3} \|\nabla \otimes u\|^2 dx ds \leq \|u_0\|_2^2, \qquad (1.22)$$

where

$$\|u(t)\|_2^2 = \sum_{k=1}^d \int_{R^3} |u_k(t,x)|^2 dx, \quad \nabla \otimes u = \sum_{k=1}^d \sum_{j=1}^d |\partial_k u_j|^2.$$

Definition 1.2. **(Mild solution)** *The Kato mild solution of (1.1), (1.2) is a solution of (1.7) constructed as a fixed point of the transform*

$$v \mapsto e^{t\Delta}u_0(x) - \int_0^t e^{(t-s)\Delta}\mathbf{P}\nabla \cdot (v \otimes v)(\theta, x)d\theta = e^{t\Delta}u_0 - B(v,v).$$
$$(1.23)$$

.

Note that the right hand side of (1.7)

$$e^{t\Delta}u_0(x) - \int_0^t e^{(t-s)\Delta}\mathbf{P}\nabla \cdot (u \otimes u)(\theta, x)d\theta = \Phi(t,x,u) \qquad (1.24)$$

is a nonlinear map in the corresponding space and the solution u is obtained by the iterative procedure

$$u^0 = e^{t\Delta}u_0, \quad u^{n+1} = e^{t\Delta}u_0 - B(u^n, u^n). \qquad (1.25)$$

Hence to construct a mild solution to (1.1), (1.2) means to find a suitable functional space for which $\Phi(t,x,u)$ given by (1.20) is a contraction.

To this end one has to find a subspace $\mathcal{E}_T$ of $L^2_{uloc,x}L^2_t((0,T) \times R^3)$ so that the bilinear transformation $B(u,v)$ of the form (1.11) is bounded as a map $\mathcal{E}_T \times \mathcal{E}_T \to \mathcal{E}_T$. Then one may consider the space $\mathbf{E}_T \subset \mathcal{S}'$ defined by $f \in \mathbf{E}$ iff $f \in \mathcal{S}'$ and $(e^{t\Delta}f)_{0<t<T} \in \mathcal{E}_T$ and prove the following result.

Theorem 1.3. The Picard contraction principle.

Let $\mathcal{E}_T \subset L^2_{uloc,x}L^2_t([0,T) \times R^3)$ be such that the bilinear map B is bounded on $\mathcal{E}_T$ Then:

(a) If $u \in \mathcal{E}_T$ is a weak solution for the Navier-Stokes equation (1.1) (1.2) then the associated initial value belongs to $\mathbf{E}_T$.

(b) There exists a positive constant C such that for all $u_0 \in \mathbf{E}_T$ satisfying $\nabla \cdot u = 0$ and $\|e^{t\Delta}u_0\|_{\mathcal{E}_T} < \infty$ there exists a weak solution $u \in \mathcal{E}_T$ of (1.1) (1.2) associated with the initial value u_0

$$u = e^{t\Delta}u_0 - \int_0^t e^{(t-s)\Delta}\mathbf{P}\nabla \cdot (u \otimes u)ds. \qquad (1.26)$$

The classical results assert that for sufficiently smooth initial data for example for u_0 in the Sobolev space H^k , $k > \frac{d}{2} + 1$, there exists a short time strong unique solution to (1.1), (1.2). On the other hand Leray proved the existence of a global weak solution of finite energy, i.e. $u \in L^2$ called the Leray-Hopf solution. Although the uniqueness and full regularity of this solution are still an open problem nevertheless one knows that if a strong solution exists then a weak solution coincides with it.

2 Probabilistic approaches to the solution of the N-S equations

Along with above functional analytical approaches recently a number of probabilistic approaches to the problems of hydrodynamics was developed (see [18]-[17], [7]). In this section we give a short survey of several different probabilistic approaches.

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space, $w(t), B(t)$ be a couple of independent Wiener processes valued in R^3.

Assume that $(u(t,x), p(t,x))$ is a unique strong solution to (1.1), (1.2) or (to be more precise) to (1.1), (1.5) and set $\nu = \frac{\sigma^2}{2}$ to simplify notations in stochastic equations. Since in this case u is C^2-smooth one can check that the stochastic equation

$$d\xi(\tau) = -u(t - \tau, \xi(\tau))d\tau + \sigma dw(\tau), \quad \xi(0) = x, \qquad (2.1)$$

has a unique solution and the function

$$v(t, x) = E[u_0(\xi(t)) - \int_0^t \nabla p(t - \tau, \xi(\tau))d\tau] \qquad (2.2)$$

satisfies (1.1) and hence equals to u by the uniqueness of the strong solution to (1.1). The relation

$$-2p(t, x) = \int_0^\infty E\gamma(t, x + B(\theta))d\theta \qquad (2.3)$$

with γ given by (1.4) allows us to verify that $\operatorname{div}u = 0$. Thus (2.2),(2.3) give the probabilistic representation of the solution to the N-S system.

If $u(t,x)$ is a C^2-smooth solution to (2.1)-(2.3), then Ito's formula yields that for $v(\theta, x) = u(t - \theta, x)$

$$v(t, \xi(t)) = v(\theta, x) + \int_\theta^t [\frac{\partial v}{\partial \tau} - (v, \nabla)v + \frac{\sigma^2}{2}\Delta v](\tau, \xi(\tau))d\tau +$$

$$\int_\theta^t \sigma \nabla v(\tau, \xi(\tau))dw(\tau).$$

Then (2.2) and the relation $u'_\tau(t - \tau, x) = -v'_\tau(\tau, x)$ yield

$$Eu(0, \xi(t)) = v(t, x) - E[\int_0^t [\frac{\partial u}{\partial \tau} + (u, \nabla)u - \frac{\sigma^2}{2}\Delta u](t - \tau, \xi_x(\tau)) +$$

$$\nabla p(t - \tau, \xi_x(\tau))]d\tau] + E[\int_0^t \nabla p(t - \tau, \xi_x(\tau))]d\tau].$$

Finally it results from (2.2) that

$$E[\int_0^t [\frac{\partial u}{\partial \tau} + (u, \nabla)u - \frac{\sigma^2}{2}\Delta u + \nabla p](t - \tau, \xi_x(\tau))]d\tau] = 0.$$

Since the latter equality holds for all t and x we deduce that (1.1)also holds. The relation $u(0, x) = u_0(x)$ immediately follows from (2.2).

The system (2.1)-(2.3) is a closed system of equations and we can try to look for its solution. Then at a second step we will look for the connection between this solution and a solution of the N-S system.

This approach was realized in paper [7]. It appears that to prove the existence of smooth solutions to (2.1)-(2.3) we have to consider the stochastic representations for ∇u and ∇p along with this system.

Using general results of diffusion process theory and in particular the Bismut-Elworthy formula [8] we note that heuristic differentiation of (2.1)- (2.3) leads to the relations

$$\nabla_k u_i(t,x) = E[\nabla_j u_{0i}(\xi(t))\eta_{jk}(t)-$$

$$\int_0^t \frac{1}{\sigma\tau}(\nabla_i p(t-\tau,\xi(\tau))\int_0^\tau \eta_{kl}(\theta)dw_l(\theta))d\tau] \qquad (2.4)$$

and

$$d\eta_{ik} = -\nabla_j u_i(t-\tau,\xi(\tau))\eta_{jk}(\tau)d\tau, \quad \eta_{ik}(0) = \delta_{ik}. \qquad (2.5)$$

In addition by Bismut-Elworthy's formula (integration by parts) we can derive from (1.17) the probabilistic representation for $\nabla p(t,x)$

$$2\nabla p(t,x) = -\int_0^\infty \frac{1}{s}E[\gamma(t,x+B(s))B(s)]ds. \qquad (2.6)$$

The main results in [7] can be stated in the following way.

Let $V = (u,\nabla u)$, $\mathcal{V}_1 = \{V(t,x) : \|V(t)\|_{L^r_{loc}} < \infty\}$, if $\frac{5}{3} < r < 2$ and $\mathcal{V}_2 = \{V(t,x) : \|V(t)\|_{L^r_{loc}} < \infty\}$, if $r > 3$, $\mathcal{V} = \mathcal{V}_1 \cap \mathcal{V}_2 \cap C^{1+\alpha}$ and let $\mathcal{M} = C([0,T],\mathcal{V})$ denote the Banach space with the norm

$$\|V\|_{r,\alpha} = sup_{t\in[0,T]}[\|V(t)\|_\mathcal{V} + [\nabla u(t)]_\alpha].$$

Theorem 2.1.([7])*Assume that $V(0) = V_0 \in \mathcal{V}$. Then there exist a bounded interval $[0,T]$ depending on V_0 and a unique solution $(\xi(t),u(t,x),p(t,x),\eta(t),\nabla u(t,x))$ to the system (2.1)-(2.5) belonging to $\mathcal{M}$ for each $\tau \in [0,T]$.*

Theorem 2.2.([7])*Assume that the conditions of theorem 2.1 hold and $u_0 \in C^{2+\alpha}$. Then there exists an interval $[0,T_1]$, $T_1 \leq T$, such that for all $t \in [0,T_1]$ there exists a unique solution to (1.1), (1.4) in $\tilde{\mathcal{M}} \subset \mathcal{M}$ where $\tilde{\mathcal{M}} = \mathcal{M}\cap C^2$ and this solution is given by (2.2),(2.3).*

A close approach based on a similar diffusion process was developed by Busnello, Flandoli, Romito [18], though their starting point was the system that governs the vorticity $v = \operatorname{curl} u$ and velocity u. The corresponding probabilistic counterpart of (1.15),(1.16) can be presented in the form of the following stochastic system

$$d\xi(\tau) = -u(t-\tau,\xi(\tau))d\tau + \sigma dw(t), \; \xi(s) = x, \qquad (2.7)$$

and the following two relations

$$v(t,x) = E[U(t,s)u_0(\xi(t)), \qquad (2.8)$$

$$2u(t,x) = \int_0^\infty \frac{1}{\theta}E[v(t,x+B(\theta)) \times B(\theta)]d\theta, \qquad (2.9)$$

where $U(t,s) = exp(\int_s^t \nabla u(t-\tau,\xi(\tau)d\tau)$ is a solution to the linear equation

$$dU_s^{t,x} = \nabla u(t-s,\xi(s))U_s^{t,x}ds, \quad U_0^{t,x} = Id. \qquad (2.10)$$

The main results in [18] are as follows. Denote by

$$\mathcal{U}^\alpha(T) =$$

$$\{u \in C([0,T], C_b^1(R^3, R^3)) \cap L^\infty([0,T], C_b^{1,\alpha}(R^3, R^3)) | \mathrm{div}\, u = 0\}$$

the Banach space endowed with the norm

$$\|u\|_{\mathcal{U}^\alpha} = \sup \mathrm{ess}_{0 \le t \le T} \|u(t)\|_{C_b^{1,\alpha}}$$

and by

$$\mathcal{V}^{\alpha,q}(T) = \{v \in C([0,T], C_b(R^3, R^3)) \cap L^\infty([0,T], C_b^\alpha(R^3, R^3))\}$$

the Banach space endowed with the norm

$$\|v\|_{\mathcal{V}^{\alpha,q}} = \sup \mathrm{ess}_{0 \le t \le T} \|v(t)\|_{L^q \cap C_b^\alpha}.$$

Theorem 2.3.([18]) *Given $p \in [1, \frac{3}{2}), \alpha \in [0,1]$ and $T > 0$ let $\xi_0 \in C_b^\alpha(R^3, R^3) \cap L^p(R^3, R^3)$ and set*

$$\varepsilon_0 = \|v_0\|_{C_b^\alpha \cap L^p}.$$

Then there exists $\tau \in [0,T]$ depending only on ε_0, such that there is a unique solution of (2.7)-(2.10). The diffusion process $\xi(t)$ plays a role of a Lagrangian path, vector field u belongs to $\mathcal{U}_\alpha$, and vector field v belongs to $\mathcal{V}^{\alpha,p}(\tau)$. In addition the deformation matrix $U_s^{x,t}$ satisfies (2.7).

After these developments P.Constantin kindly attracted our attention to his papers [11] [12] where the Lagrangian approach was successfully applied to the investigation of the Navier-Stokes system. The presentation of this approach and the discussion of their similarity and difference will be given in the last section of the present article.

A probabilistic representation of the solution to the Fourier transformed Navier-Stokes equation (1.11) was constructed by Le Jan and Sznitman [16].

To describe their approach recall the definition of the solution of the Fourier representation (FNS) of the Navier-Stokes system.

First for a solution $u(t,x)$ of (1.26) and its Fourier transform $\hat{u}(t,\lambda)$ one can introduce a function $\chi(t,\lambda)$ defined on $[0,T] \times R^3$ such that

$$\chi_t(\lambda) = \frac{2}{\nu}(\frac{\pi}{2})^{\frac{3}{2}} |\lambda|^2 \hat{u}_t(\lambda), \text{ a.e. for } t \in [0,T],$$

and

$$\chi_t(\lambda) \cdot \lambda = 0, \quad \chi_t(-\lambda) = \bar{\chi}_t(\lambda).$$

In addition, for Lebesgue a.e. λ, $\chi_t(\lambda)$ solves the equation

$$\chi_t(\lambda) = exp(-\nu|\lambda|^2 t)\chi_0(\lambda) + \int_0^t \nu|\lambda|^2 e^{-\nu|\lambda|^2(t-s)} \frac{1}{2}[\chi_s \circ \chi_s(\lambda)]ds, \tag{2.11}$$

where

$$\chi_1 \circ \chi_2(\lambda) = -\frac{i}{\pi^3} \int (\chi_1(\lambda_1) \cdot e_\lambda) \Pi(\lambda)\chi_2(\lambda - \lambda_1)\frac{|\lambda|d\lambda_1}{|\lambda_1|^2|\lambda - \lambda_1|^2},$$

13

if the initial function χ_0 is a measurable function and for a.e. $\lambda \in R^3\backslash\{0\}$

$$\chi_0 : R^3\backslash\{0\} \to C^3, \quad \chi_0(\lambda) \cdot \lambda = 0, \quad \chi_0(-\lambda) = \overline{\chi_0(\lambda)}.$$

Introducing the kernel K from $R^3\backslash 0$ to $(R^3\backslash 0)^2$

$$\int h(\lambda_1, \lambda_2) K_\lambda(d\lambda_1, d\lambda_2) = \frac{1}{\pi^3} \int h(\lambda_1, \lambda - \lambda_1) \frac{|\lambda| d\lambda_1}{|\lambda_1|^2 |\lambda - \lambda_1|^2}$$

for $h \geq 0$ measurable on $(R^3\backslash 0)^2$ one gets

$$\chi_1 \circ \chi_2(\lambda) = -i \int (\chi_1(\lambda_1) \cdot e_\lambda) \Pi(\lambda) \chi_2(\lambda_2) K_\lambda(d\lambda_1, d\lambda_2).$$

It turns out that K is a Markovian kernel with some remarkable features that allow to study existence and uniqueness problems for (2.11) with the help of a critical branching process on $R^3\backslash\{0\}$ called the stochastic cascade. Namely, LeJan and Sznitman have described a particle located in λ such that after an exponentially holding time with parameter $\nu|\lambda|^2$ with equal probability the particle either dies or gives birth to two descendants, distributed according to K_λ. A representation formula for the solution of (2.11) is constructed as the expectation of the result of a certain operation performed along the branching tree generated by the stochastic cascade.

One more probabilistic model was recently developed by M. Ossiander [17]. A binary branching process with jumps that corresponds to the formulation of solutions to N-S in physical space was constructed in [17].

Once again the N-S system is reformulated incorporating incompressibility via the Leray projection $\mathbf{P}$ and then the Duhamel principle is applied to derive

$$u = e^{\nu t \Delta} u_0 - \int_0^t e^{-\nu(t-s)\Delta} \mathbf{P} \nabla \cdot (u \otimes u)(s) ds + \int_0^t e^{-\nu(t-s)\Delta} \mathcal{P} f(s) ds, \tag{2.12}$$

$$\nabla \cdot u_0 = 0. \tag{2.13}$$

Let $K(y,t) = (2\pi t)^{-\frac{3}{2}} e^{-\frac{|y|^2}{2t}}$ be the transition density of the Brownian motion $w(t) \in R^3$

$$(\mathbf{P}_y)_{ij} = \delta_{ij} - (e_y)_i (e_y)_j$$

$$b_1(y; u, v) = (u \cdot e_y)\mathbf{P}_y v + (v \cdot e_y)\mathbf{P}_y u,$$

$$b_2(y; u, v) = b_1(y; u, v) + u \cdot (I - 3 e_y e_y^T) v e_y.$$

Then (2.12) can be rewritten in the form

$$u(x,t) = \int_{R^3} u_0(x - y) K(y, 2\nu t) dy +$$

$$\int_0^t \int_{R^3} \{\frac{|z|}{4\nu s} K(z, 2\nu s) b_1(z; u(x - z, t - s), u(x - z, t - s)) +$$

$$(\frac{1}{|z|} K(z, 2\nu s) - \frac{3}{4\pi |z|^4} \int_{\{y:|y| \le z\}} K(z, 2\nu s) dy)$$

$$b_2(z; u(x - z, t - s), u(x - z, t - s)) + (K(z, 2\nu s) \mathbf{P}_z -$$

$$\frac{1}{4\pi |z|^3} (I - 3 e_z e_z^T) \int_{\{y:|y| \le z\}} K(y, 2\nu s) dy) g(x - z, t - s) \} dz ds. \quad (2.14)$$

Theorem 2.4. ([17]) *Let $h : R^3 \to [0, \infty]$ and $\tilde{h} : R^3 \to [0, \infty]$ with h locally integrable and $h, \tilde{h}$ jointly satisfying*

$$\int_{R^3} h^2(x - y)|y|^{-2} dy \le h(x) \, and \int_{R^3} \tilde{h}(x - y)|y|^{-1} dy \le h(x)$$

for all $x \in R^3$. If for all $x \in R^3$ and $t > 0$

$$(4\pi \nu t)^{-\frac{3}{2}} | \int_{R^3} u_0(x - y) e^{-\frac{|y|^2}{4\nu t}} dy | \le \pi \nu \frac{h(x)}{11}$$

and

$$|g(x, t)| < (\pi \nu)^2 \frac{\tilde{h}(x)}{11}$$

then there exists a collection of probabilistic measures $\{P_x : x \in R^3\}$ defined on a common measurable space $(\Omega, \mathcal{F})$ and a measurable function $\Sigma : (0, \infty) \times \Omega \to R^3$ such that

$$P_x(\{\omega : |\Sigma(t, \omega)| < \frac{2\pi \nu}{11} \, for \, all \, t > 0) = 1$$

for which a weak solution $u(t, x)$ to the N-S can be presented in the form

$$u(x, t) = h(x) \int_{\Omega} \Sigma(t, \omega) dP_x(\omega) \quad for \, all x \in R^3, t > 0.$$

Furthermore the solution u is unique in the class

$$\{v \in (S'(R^3 \times (0, \infty)))^3 : |v(t, x)| < \frac{2\pi \nu h(x)}{11}\}$$

for all $x \in R^3, t > 0$.

Our survey is still far from being exhaustive. As already mentioned the discussion of the Euler-Lagrangian approach developed by Constantin and Iyer will be postponed to of the present paper the last section.

3 A probabilistic representation of the solution to the Poisson equation

Within the framework of the approach developed in this paper we intend to construct diffusion processes associated with the system (1.1) (1.5). First we will start with (1.5) and recall some results concerning the solution of the Poisson equation in an open domain $G \subseteq R^3$.

First we recall that by the divergence theorem a $C^2(G)$ solution of $-\Delta p = \gamma$ satisfies the integral identity

$$\int_G \nabla p \cdot \nabla \phi \, dx = -\int_G \gamma \phi \, dx$$

for all $\phi \in C_0^1(G)$. In the space $W_0^{1,2}(G)$ which is the completion of $C_0^1(G)$ under the inner product

$$\langle p, \phi \rangle = \int_G \nabla p \cdot \nabla \phi \, dx$$

the linear functional

$$F(\phi) = -\int_G \gamma \phi \, dx$$

may be extended to a bounded linear functional on the space $W_0^{1,2}(G)$. Hence by the Riesz theorem there exists an element $p \in W_0^{1,2}(G)$ satisfying $\langle p, \phi \rangle = F(\phi)$ for all $\phi \in C_0^1(G)$. Then the existence of a generalized solution to the Dirichlet problem $-\Delta p = \gamma$ and $p = 0$ on ∂G is readily established. The question of classical existence is accordingly transformed into the question of regularity of generalized solution under the appropriately smooth bounded conditions.

We give in this section a brief summary of a probabilistic approach to the solution of the Poisson equation. We will try to give the probabilistic proofs of the necessary facts inasmuch as they are known. Proofs of similar statements can be found in [18]. The source for analytical results is the book by Gilbarg and Trudinger [22].

Consider the Poisson equation

$$-\Delta p(x) = \gamma(x) \tag{3.1}$$

where p and γ are scalar integrable functions defined on G. A Newton potential with density γ is defined by

$$N\gamma(x) = \frac{1}{4\pi} \int_G \frac{1}{\|x - y\|} \gamma(y) dy. \tag{3.2}$$

If γ is regular and has a compact support then $N\gamma$ is known to be a solution of the Poisson equation (2.1).

To derive a probabilistic interpretation of the relation (3.2) we consider the generator $\mathcal{A} = \frac{1}{2}\Delta$ of a Wiener process $B(t) \in R^3$ defined on a given probability space $(\Omega, \mathcal{F}, P)$. It is well known that on the space $C_0(R^3)$ of all continuous functions vanishing at infinity the Wiener process generates the strongly continuous semigroup

$$T_t\gamma(x) = E\gamma(x + B(t)), \quad x \in R^3, t \geq 0, \gamma \in C_0(R^3).$$

Given a function with a compact support in G we extend it to the whole space R^3 by zero.

By a direct computation we can check that

$$\int_0^\infty E[\gamma(x + B(t))]dt = \int_{R^3} \gamma(x + y) \int_0^\infty \frac{1}{(2\pi t)^{\frac{3}{2}}} e^{-\frac{1}{2t}\|y\|^2} dt dy =$$

16

$$\int_{R^3} \frac{1}{2\pi \|y\|} \gamma(x+y) dy = 2N\gamma. \tag{3.3}$$

To prove that $p = 2N\gamma$ solves the Poisson equation

$$-\Delta p = \gamma$$

we need some additional regularity properties of $N\gamma$.

Lemma 3.1. Let $\gamma \in L^m(R^3) \cap L^q(R^3)$ with $1 \le m < 3 < q < \infty$. Then $N\gamma \in C_0(R^3)$ and

$$\|N\gamma\|_\infty \le C_{m,q}(\|\gamma\|_m + \|\gamma\|_q).$$

Proof. First we note that for every l, m such that $\frac{1}{l} + \frac{1}{r} = 1$ by Hölder inequality we have

$$E|\gamma(x+B(t))| = \frac{1}{(2\pi t)^{\frac{3}{2}}} \int_{R^3} |\gamma(x+y)| e^{-\frac{\|y\|^2}{2t}} dy \le \tag{3.4}$$

$$C_r t^{-\frac{3}{2}+\frac{3}{2l}} \|\gamma\|_r \le C_r t^{-\frac{3}{2r}} \|\gamma\|_m,$$

since $-\frac{3}{2} + \frac{3}{2l} = -\frac{3}{2r}$.

Finally we rewrite the left hand side of (3.1) as

$$\int_0^\infty E[\gamma(x+B(t))]dt = \int_0^1 E[\gamma(x+B(t))]dt + \int_1^\infty E[\gamma(x+B(t))]dt$$

and applying the estimate (3.4) for $r = q$ and $r = m$ we derive

$$\int_0^\infty E[\gamma(x+B(t))]dt \le C(\|\gamma\|_m + \|\gamma\|_q),$$

with $C = max(C_m, C_q)$.

By Sobolev embeddings it is known [22] that if $\gamma \in L^1(R^3)$ then $N\gamma \in C(R^3)$. To check that $N\gamma \in C_0(R^3)$ we note that for any $R > 0$ we can rewrite the left hand side of (3.3) as

$$\int_0^\infty E[\gamma(x+B(t))]dt = \int_0^\infty E[\gamma(x+B(t))I_{\|B(t)\|>R}]dt+ \tag{3.5}$$

$$\int_0^\infty E[\gamma(x+B(t))I_{\|B(t)\|\le R}]dt.$$

Let us prove that the first term on the right hand side of (3.5) converges to 0 uniformly in x as $R \to \infty$ and the second term converges to 0 as $\|x\| \to \infty$ for each R. For the first term we apply the estimate (3.4) to derive

$$\sup_{x\in R^3} E[|\gamma(x+B(t))|I_{\{\|B(t)\|>R\}}] \le$$

$$C(\|\gamma\|_p + \|\gamma\|_q)(t^{-\frac{3}{2m}}I_{\{[1,\infty)\}}(t) + t^{-\frac{3}{2q}}I_{\{[0,1)\}}(t))$$

and

$$\sup_{x\in R^3} E[|\gamma(x+B(t))|I_{\{\|B(t)\|>R\}}] \le Ct^{-\frac{3}{2}}\|\gamma\|_m (\int_{\{\|y\|>R\}} e^{-\frac{\|y\|^2}{2t}} dy)^{\frac{1}{q}} \to 0$$

17

as $R \to \infty$. To obtain the estimate for the second term we apply (3.4) once again and obtain

$$E[|\gamma(x + B(t))|I_{\|B(t)\| \le R}] \le$$

$$Ct^{-\frac{3}{2m}}\left(\int_{R^3} |\gamma(y)|^p I_{\{\|y-x\| \le R\}} dy\right)^{\frac{1}{m}} I_{\{[1,\infty)\}}(t) +$$

$$Ct^{-\frac{3}{2q}}\left(\int_{R^3} |\gamma(y)|^q I_{\{\|y-x\| \le R\}} dy\right)^{\frac{1}{q}} I_{[0,1)}(t)),$$

that yields after the integration in time that the second term on the right hand side of (3.5) converges to 0, since $\gamma \in L^m(G) \cap L^q(G)$ and is zero outside G.

To study derivatives of $N\gamma$ we apply the Bismut-Elworthy-Li formula

$$\nabla_{x_i} E[\gamma(x + B(t))] = \frac{1}{t} E[\gamma(x + B(t))B_i(t)]$$

that holds for a regular γ.

Lemma 3.2. Let $\gamma \in L^m(R^3) \cap L^q(^3)$ for some $1 \le m < \frac{3}{2} < 3 < q < \infty$. Then $\nabla N\gamma \in C_0(R^3)$ and for each $x \in R^3$

$$2\nabla_{x_i} N\gamma(x) = \int_0^\infty \frac{1}{t} E[\gamma(x + B(t))B_i(t)]dt, \quad i = 1, 2, 3. \qquad (3.6)$$

Moreover

$$\|\nabla N\gamma\|_\infty \le C_{mq}(\|\gamma\|_m + \|\gamma\|_q). \qquad (3.7)$$

Proof. By the Hölder inequality

$$\frac{1}{t} E|\gamma(x + B(t))B_i(t)|dt = \frac{C}{t^{\frac{5}{2}}} \int_{R^3} |\gamma(x + y)y_i|e^{-\frac{\|y\|^2}{2t}} dy \le \qquad (3.8)$$

$$\frac{C}{t^{\frac{5}{2}}} \|\gamma\|_m t^{\frac{1}{2} + \frac{3}{2q}} = C_m \|\gamma\|_m t^{-\frac{1}{2} - \frac{3}{2q}}.$$

Finally to give the sup estimate for the second derivative of $N\gamma$ one has to apply the Schauder estimates and the Bismut-Elworthy-Li formula.

Let us recall two more useful results (see [22] theorem 4.5) concerning the Newton potential.

Lemma 3.3. Let $\gamma \in L^q(R^3) \cap C_b^\alpha(R^3)$ with $1 \le q \le \frac{3}{2}$. Then $N\gamma \in C_b^{2,\alpha}(R^3) \cap C_0(R^3)$,

$$\|N\gamma\|_{C_b^{2,\alpha}(R^3)} \le C(\|\gamma\|_{L^q(R^3)} + \|\gamma\|_{C_b^\alpha(G)})$$

and $p = 2N\gamma$ is the unique solution of the Poisson equation

$$-\Delta p = \gamma$$

in $C_0(R^3) \cap C^2(R^3)$.

Theorem 3.4. Let $N\gamma \in C_0^2(R^3), \gamma \in C_0^2(R^3)$ satisfy the Poisson equation $\Delta N\gamma = \gamma$ in R^3. Then $N\gamma \in R^3$ and if $B = B_R(x_0)$ is any ball containing the support of $N\gamma$ then

$$\|\nabla^2 N\gamma\|_{0,\alpha;B} \le C_\alpha \|\gamma\|_{0,\alpha;B}, \quad \|N\gamma\|'_{1,B} \le CR^2 \|\gamma\|_{0,B}. \qquad (3.9)$$

In the sequel we will need as well L_q type estimates for the Newtonian potential.

Lemma 3.5. *The operator N maps $L^q(R^3)$ into $L^q(R^3)$ and there exists a positive constant C such that*

$$\|N\gamma\|_{L^q(R^3)} \leq C\|\gamma\|_{L^q(R^3)} \tag{3.10}$$

Proof. By Hölder inequality we have

$$|N\gamma|(x) = |\int_G \gamma(y)(\Gamma(x-y))^{\frac{1}{q}}(\Gamma(x-y))^{1-\frac{1}{q}}dy| \leq$$

$$\{\int_G |\gamma(y)|^q \Gamma(x-y)dy\}^{\frac{1}{q}}\{\int_G \Gamma(x-y)dy\}^{1-\frac{1}{q}} \leq$$

$$C\{\int_G |\gamma(y)|^q \Gamma(x-y)dy\}^{\frac{1}{q}}.$$

Next we obtain by Fubini's theorem

$$\int_{R^3} |N\gamma|^p(x)dx \leq \int_{R^3} C^p \{\int_{R^3} |\gamma(y)|^p \Gamma(x-y)dy\}dx =$$

$$C^p \int_G \int_{R^3} |\gamma(y)|^p \Gamma(x-y)dydx = C^p \int_{R^3} |\gamma(y)|^p (\int_{R^3} \Gamma(x-y)dx)dy \leq$$

$$C_1 \int_{R^3} |\gamma(y)|^p dy.$$

Note that all above results in this section are valid if we consider a bounded domain $G \subset R^3$ instead of R^3. To get further regularity properties of the Newton potential we need more auxiliary results.

Define the distribution $\nu_\gamma(\lambda)$ of the function $\gamma : G \to R^1$ by

$$\nu_\gamma(\lambda) = |\{x \in G : |\gamma(x)| > \lambda\}| \tag{3.11}$$

where $|G|$ denotes the Lebesgue volume of the domain G.

Lemma 3.6. *Assume that $\gamma \in L^q(G)$ for some $q > 0$. Then*

$$\nu_\gamma(\lambda) \leq \lambda^{-q} \int_G |\gamma(x)|dx,$$

$$\int_G |\gamma(x)|^q dx = p \int_0^\infty \lambda^{q-1} \nu_\gamma(\lambda)d\lambda.$$

Proof. It is easy to check that

$$\int_G |\gamma(x)|^p dx \geq \int_{\{\gamma > \lambda\}} |\gamma(x)|^p dx \geq \lambda^p |\{x : \gamma(x) > \lambda\}| = \lambda^p \nu_\gamma(\lambda).$$

If $p = 1$ we can apply the Fubini theorem to change the order of integration

$$\int_G |\gamma(x)|dx = \int_G \int_0^{|\gamma(x)|} dtdx = \int_0^\infty \int_G I_{\{x \in G : \gamma(x) > \lambda\}}dxd\lambda =$$

$$\int_0^\infty \nu_f(\lambda)d\lambda.$$

19

For arbitrary q we have

$$\nu_{\gamma^q}(\lambda) = |\{x : \gamma(x) > \lambda^{\frac{1}{q}}\}| = \nu_\gamma(\lambda^{\frac{1}{q}})$$

and hence

$$p \int_0^\infty \lambda^{q-1} \nu_\gamma(\lambda) d\lambda = \int_0^\infty \nu_{\gamma^q}(\lambda^q) d(\lambda^q) = \int_G |\gamma(x)|^q dx.$$

Lemma 3.7. *Let $\gamma \in L^q(G)$ for some $1 < q < \infty$. Then $N\gamma \in W^{1,q}(G)$ and*

$$\|\nabla^2 N\gamma\|_{L^q(G)} \le C(q,G) \|\gamma\|_{L^q(G)} \tag{3.12}$$

Moreover for $q = 2$ the equality

$$\int_{R^3} \|\nabla^2 N\gamma\|^2(x) dx = \int_G \gamma^2(x) dx \tag{3.13}$$

holds.

Proof. The proof of this fact is based on the Calderon-Zygmund technique of cube decomposition and estimates of the function $\nu_\gamma(\lambda)$ of the form (2.5).

Let $\tilde{K}$ be a cube in R^3, $\gamma \ge 0$ integrable, and finally fix $\kappa > 0$ such that

$$\frac{1}{|\tilde{K}|} \int_{\tilde{K}} \gamma(x) dx \le \kappa.$$

Bisect $\tilde{K}$ into 2^3 equal (in volume) subcubes. Let Q be a set of those subcubes K for which $\frac{1}{|K|} \int_K \gamma(x) dx > \kappa$. For each of the remaining subcubes (which do not belong to Q) we repeat the same procedure, that is bisect each one into 2^3 sub-cubes and add those smaller ones, where f is highly concentrated to Q. Now repeating the procedure again and again we obtain a partition of $\tilde{K}$. For any K in Q denote by $\hat{K}$ its immediate predecessor. Since $K \in Q$, while $\hat{K} \notin Q$, we have

$$\lambda < \frac{1}{|K|} \int_K \gamma(x) dx < \frac{1}{|K|} \int_{\hat{K}} \gamma(x) dx = \frac{|\hat{K}|}{|K|} \frac{1}{|\hat{K}|} \int_{\hat{K}} \gamma(x) dx < 2^3 \lambda.$$

Set $F = \cup_{K \in Q} K$, $J = \tilde{K} \backslash F = \cap_{K \in Q} K^C$. Note that each point in J belongs to infinitely many nested cubes with bounded concentration of γ with diameters converging to zero, that is $\frac{1}{|K_i|} \int_{\hat{K}_i} \gamma(x) dx \le \kappa$, with $|K_i| \to 0$. By the Lebesgue theorem we deduce that $\frac{1}{|K_i|} \int_{\hat{K}_i} \gamma(x) dx \to \gamma$ a.e. with respect to the Lebesgue measure, that is $\gamma \le \kappa$ a.e. on J. Then we have an average estimate on F and a point-wise estimate on J.

At the second step we need the Marcinkiewicz interpolation theorem.

Marcinkiewicz interpolation theorem. *Let $1 \le q < r < \infty$ and let $\mathcal{T} : L^q(G) \cap L^r(G) \to L^q(G) \cap L^r(G)$ be a linear map. Suppose there exist constants C_1, C_2 such that $\forall \gamma \in L^q(G) \cap L^r(G)$ and for any $\lambda > 0$*

$$\nu_{\mathcal{T}\gamma}(\lambda) \le \left(\frac{C_1 \|\gamma\|_{L^q(G)}}{\lambda} \right)^q, \quad \nu_{Q\gamma}(\lambda) \le \left(\frac{C_2 \|\gamma\|_{L^r(G)}}{\lambda} \right)^r.$$

Then for any exponent m such that $q < m < r$ the map $\mathcal{T}$ can be extended to a map from $L^m(G)$ to $L^m(G)$ and

$$\|\mathcal{T}\gamma\|_{L^m(G)} \le KC_1^\alpha C_2^{1-\alpha}\|\gamma\|_{L^m(G)}.$$

all $\gamma \in L^q(G) \cap L^p(G)$ where $\frac{1}{m} = \frac{\alpha}{q} + \frac{1-\alpha}{r}$ and the constant K depends only on m, q and r.

At the end we define an operator $\mathcal{T} : L^2(G) \to L^2(G)$ by $\mathcal{T}\gamma = \nabla_i \nabla_j N\gamma$ to obtain the necessary result.

Theorem 3.8.(Calderon-Zygmund inequality) *Let $\gamma \in L^p(G)$, $1 < q < \infty$. Then the Newton potential $N\gamma = p \in W^{2,q}(G)$, solves the Poisson equation $\Delta p = \gamma$ a.e. and*

$$\|\nabla^2 p\|_{L^q(G)} \le C\|\gamma\|_{L^q(G)}, \tag{3.14}$$

where C depends only on d and q. Furthermore , when $q = 2$ we have

$$\int_{R^3} \|\nabla^2 N\gamma(x)\|^2 dx = \int_G \gamma^2(x)dx.$$

For the proof of the above interpolation theorem and theorem 3.8 see, e.g., [22].

4   Probabilistic representations of weak solutions of parabolic equations

In this section we adapt the results of the Kunita theory of stochastic flows acting on Schwartz distributions [19],[20] to the case under consideration. The considerations in this section are similar to [21].

Unlike the Kunita case we assume here that the coefficients of SDEs under consideration are at most $C^{1+\alpha}$ -smooth with $0 < \alpha < 1$, but on the other hand it is enough for our present purpose to restrict ourself to nonsingular initial data for the Cauchy problem for parabolic equations and hence we consider stochastic flows in Sobolev spaces $\mathcal{H}^k$ for $k = 1, -1$.

It is more convenient for computational reasons to use sometimes the Stratonovich form of the Ito equation. Recall that a process $\xi(t)$ having the Ito differential of the form

$$d\xi(t) = [a(\xi(t)) + \frac{1}{2}Tr\nabla\sigma(\xi(t))\sigma(\xi(t))]dt + \sigma(\xi(t))dw$$

has the Stratonovich differential of the form

$$d^S\xi(t) = a(\xi(t))dt + \sigma(\xi(t)) \circ dw.$$

We say that condition **C 4.1** holds if for all $t \in [0, T]$ the functions $g(t)$ and σ belongs respectively to $C_b^{1+\alpha}$ and $C_b^{2+\alpha}$.

Throughout this section we assume that **C4.1** holds. We shall first give a brief review of the results which will be needed in the sequel.

Consider a stochastic differential equation in the Stratonovich form

$$d\xi(\tau) = -g(t - \tau, \xi(\tau))d\tau - \sigma(\xi(\tau)) \circ dw(\tau), \quad \xi(s) = x \in R^3, \tag{4.1}$$

$0 \leq s \leq \tau \leq t$. Here $g(t,x) \in R^3$, $\sigma(t,x) \in R^3 \times R^3$ and $w(t) \in R^3$ is a Wiener process.

Assuming that $g(t) \in \mathbf{C^1}(R^3)$ and $\sigma(t)$ is a C^2-smooth matrix we are in the framework of the Kunita theory [6] and know that there exists a local C^1-diffeomorphism of R^3 generated by the solution $\xi_{s,x}(\tau)$ of (4.1).

Namely, by general results on the SDE theory the existence and uniqueness of the solution $\xi_{s,x}(\tau)$ to (4.1) are granted for a C^1- smooth bounded function g. Moreover, in this case, one can prove that the solution $\xi_{s,x}^g(\tau)$ of (4.1) has a modification $\phi_{s,\tau}^g(x,\omega)$ such that for all ω outside a null set $\mathcal{N} \subset \Omega$

1)$\phi_{s,\tau}^g(x,\omega)$ is continuous in (s,τ,x), and differentiable in x;

2) $\phi_{\tau,\tau}^g(\phi_{s,\tau}^g(x,\omega),\omega) = \phi_{s,\tau}^g(x,\omega)$, if $0 < s < \tau < t$;

3) the mapping $\phi_{s,\tau}^g(\omega) : R^3 \to R^3$ is a C^1- diffeomorphism in R^3.

The map $\phi_{s,\tau}^g(\omega)$ is called a stochastic flow of C^1- diffeomorphisms in R^3.

We will denote by $(\phi_{s,\tau}^g)^{-1}(\omega) = \psi_{\tau,s}^g(\omega)$ the map inverse to the stochastic flow $\phi_{s,\tau}^g(\omega)$ and will write simply $\psi_{\tau,s}^g(x)$ for $\psi_{\tau,s}^g(x,\omega)$. We check a simple property of an inverse stochastic flow.

Lemma 4.1. *Consider the σ-algebras*

$$\mathcal{F}_s^w = \sigma\{w(\theta) : \theta \in [0,s]\} \quad \mathcal{F}_{t,s}^{\hat{w}} = \sigma\{\hat{w}(\tau) - \hat{w}(\tau_1) : s \leq \tau_1 \leq \tau \leq t\}$$

and a continuous bounded process $m(s)$ adapted to $\mathcal{F}_s^w$. Then the process $f(s) = g(t-s)$ for $s \in [0,t]$ is $\mathcal{F}_{t,s}^{\hat{w}}$ adapted and for all α, β such that $0 \leq \alpha \leq \beta \leq t$ we have

$$\int_\alpha^\beta f(\tau)dw(\tau) = \int_{t-\alpha}^{t-\beta} g(s)d\hat{w}(s).$$

Proof. Note that since $\hat{w}(s) = w(t-s) - w(t)$ we have $\hat{w}(\beta) - \hat{w}(\alpha) = w(t-\beta) - w(t-\alpha)$, that yields $\mathcal{F}_{t-s}^w = \mathcal{F}_{t,s}^{\hat{w}}$.

Now we consider a partition of the interval $[0,t]$

$$\{0 = t_0 \leq t_1 \leq \ldots \leq t_k \leq t_{k+1} \leq \ldots \leq t_N = t\}$$

such that $|t_{k+1} - t_k| \to 0$ as $N \to \infty$. Set $\theta_k = t - t_k$ for $k = 1, \ldots, N$, then

$$\int_\alpha^\beta f(s)dw(s) = \lim_{n\to\infty} \sum_{k=1}^N f(t_k)[w(t_{k+1}) - w(t_k)] =$$

$$\lim_{n\to\infty} \sum_{k=1}^N g(t - \theta_k)[w(t - \theta_{k+1}) - w(t - \theta_k)] =$$

$$-\lim_{n\to\infty} \sum_{k=1}^N g(\theta_k)[\hat{w}(\theta_{k+1}) - \hat{w}(\theta_k)] = -\int_{t-\beta}^{t-\alpha} g(s)d\hat{w}(s).$$

The main point of Kunita's theory is that the stochastic flow is a bijection and that the inverse stochastic flow satisfies a couple of SDEs which will be used for different purposes. One of these SDEs is given by the following lemma due to Malliavin (see ([23], lemma 5.2.2) or [5]).

Lemma 4.2. *Let $\xi^g(\tau, x, w)$ be a solution of the stochastic equation (4.1) with $s = 0$ or equivalently of the SDE*

$$d\xi(\tau) = -g(t - \tau, \xi(\tau))d\tau + m(\xi(\tau))d\tau - \sigma(\xi(\tau))dw(\tau), \quad \xi(s) = x \tag{4.2}$$

where $m(x) = \frac{1}{2}Tr\nabla\sigma(x)\sigma(x)$. Then, for every fixed $T > 0$ we have

$$\xi(t - \theta, x, w) = \hat{\xi}(\theta, \xi(t, x, w), \hat{w})$$

for every $0 \leq \theta \leq t$, and x, a.s. (P^w).

In what follows we need as well some generalizations of the Itô formula. The first one called the Itô -Wentzel formula reads as follows.

Lemma 4.3. (Itô-Wentzel formula) *Assume that the process $\xi(t) \in R^3$ has a stochastic differential of the form*

$$d\xi(t) = g(t, \xi(t))dt + \sigma(\xi(t))dw(t)$$

and the process $f(t, x) \in R^3$ has a stochastic differential

$$df(t, x) = \Psi(t, x)dt + \Phi(t, x)dw(t)$$

with the same Wiener process $w(t)$. Let the vector field $\Psi(t, x) \in R^3$ and the operator field $\Phi(t, x) \in R^3 \times R^3$ be C^2 smooth in x and continuous in t. Then the process $\eta(t) = f(t, \xi(t))$ has a stochastic differential

$$df_m(t, \xi(t)) = \Psi_m(t, \xi(t))dt + \Phi_{mk}(t, \xi(t))dw_k + \nabla_i f_m(t, \xi(t))d\xi_i(t)+ \tag{4.3}$$

$$\frac{1}{2}\nabla_i\nabla_j f_m(t, \xi(t))\sigma_{ik}(\xi(t))\sigma_{jk}(\xi(t))dt + \nabla_i\Phi_{mk}(t, \xi(t))\sigma_{ik}(\xi(t))dt.$$

Remark 4.4. Note that (4.3) can be rewritten in the Stratonovich form as follows

$$df_m(t, \xi(t)) = \Psi_m(t, \xi(t))dt + \Phi_{mk}(t, \xi(t))dw_k + \nabla_i f_m(t, \xi(t)) \circ d\xi_i(t)+ \tag{4.4}$$

$$+\nabla_i\Phi_{mk}(t, \xi(t))\sigma_{ik}(\xi(t))dt.$$

We apply lemma 4.3 to check that the inverse flow $\psi^g_{t,0}$ to the flow $\phi^g(0, t)$ (generated by the solution of the equation in (4.1)) can also be represented as a solution of the following stochastic equation

$$d\psi^g_{t,0}(x) = \nabla\phi^g_{0,t}(\psi^g_{t,0})^{-1}g(t, x)dt + \nabla\phi^g_{0,t}(\psi^g_{t,0})^{-1}\sigma(x) \circ dw, \tag{4.5}$$

where $(\nabla\phi^g_{0,t})^{-1}$ is the inverse matrix of the Jacobian matrix $\nabla\phi^g_{0,t}$ of the map $\phi^g_{0,t}$.

Namely, we have the following statement proved by Kunita (see [6] Theorem 4.2.2) in a slightly different context.

Theorem 4.5. *Let* **C4.1** *hold and $\phi^g_{0,t}$ be the solution of the equation (4.1). Then the inverse flow $[\phi^g_{0,t}]^{-1} = \psi^g_{t,0}$ satisfies (4.5).*

Proof. To verify the statement of the theorem note first that the Jacobian matrix $\kappa^g(t) = \nabla\phi^g_{0,t}$ solves the Cauchy problem for the stochastic equation

$$d\kappa(\tau) = -\nabla g(\tau, \phi^g_{0,\tau}(y))\kappa(\tau)d\tau - \nabla\sigma(\phi^g_{0,\tau}(y))\kappa(\tau) \circ dw(\tau), \quad \kappa(0) = I. \tag{4.6}$$

23

Then, consider the stochastic process

$$G(y,t) = \int_0^t \nabla \phi_{0,\tau}^g(y)^{-1} g(\tau, \phi_{0,\tau}^g(y)) d\tau +$$

$$\int_0^t \nabla \phi_{0,\tau}^g(y)^{-1} \sigma(\phi_{0,\tau}^g(y)) \circ dw(\tau),$$

and compute $\phi_{0,t}^g(\psi_{t,0}^g(x))$, where $\psi_{t,0}^g$ has the stochastic differential

$$d\psi_{t,0}^g = dG(\psi_{t,0}^g(x), t).$$

Set $\phi_{0,t}^g(y) = \phi^g(y,t)$. By the Itô-Wentzell formula we have

$$\phi_{0,t}^g(\psi_{t,0}^g(x)) = x + \int_0^t d^S \phi^g(\psi_{\theta,0}^g(x), \theta) + \int_0^t \nabla \phi^g(\psi_{\theta,0}^g(x), \theta) \circ d\psi_{\theta,0}^g(x) =$$

$$x - \int_0^t g(\theta, \phi_{0,\theta}^g(\psi_{\theta,0}^g(x), \theta)) d\theta - \int_0^t \sigma(\phi_{0,\theta}^g(\psi_{\theta,0}^g(x), \theta)) \circ dw(\theta) +$$

$$\int_0^t \nabla \phi^g(\psi_{\theta,0}^g(x), \theta) [\nabla \phi^g(\psi_{\theta,0}^g(x), \theta)]^{-1} g(\theta, \phi_{0,\theta}^g(\psi_{\theta,0}^g(x))) d\theta +$$

$$\int_0^t \nabla \phi^g(\psi_{\theta,0}^g(x), \theta) [\nabla \phi^g(\psi_{\theta,0}^g(x), \theta)]^{-1} \sigma(\phi_{0,\theta}^g(\psi_{\theta,0}^g(x), \theta)) \circ dw(\theta) = x.$$

Hence, $\phi_{0,t}^g(\psi_{t,0}^g(x)) = x$ and thus $\psi_{t,0}^g$ is the inverse to $\phi_{0,t}^g$.

Remark 4.6. Recall that by lemma 4.2 the process $\psi_{t,0}(x) = \hat{\xi}(t)$ along with (4.5) satisfies the SDE

$$d\hat{\xi}(\theta) = g(\theta, \hat{\xi}(\theta))) d\theta + \sigma(\hat{\xi}(\theta)) \circ d\hat{w}(\theta). \qquad (4.7)$$

Denote by $J_{0,t}^g(\omega)$ the Jacobian of the map $\phi_{0,t}^g(\omega)$. Given $h \in \mathcal{H}^1$ and $f \in \mathcal{H}^{-1}$ one can define the composition of f with the stochastic flow $\psi_{t,0}^g(x)$ as a random variable with values in $\mathcal{H}^{-1}$ defined by the relation

$$\langle S_{t,0}^g(\omega), h \rangle = \langle f, h \circ \phi_{0,t}^g(\omega) J_{0,t}^g(\omega) \rangle, \quad h \in \mathcal{H}^1, \qquad (4.8)$$

for any t and $\omega \notin \mathcal{N}$. Note that if $\tilde{f}$ is a distribution of the form $\tilde{f} = f(x) dx$ where f is a continuous function then $\tilde{f} \circ \psi_{t,0}$ is just the composition of the function f with the map $\psi_{t,0}(\omega)$ and

$$\int_{R^3} f(\psi_{t,0}(y,\omega)) h(y) dy = \int_{R^3} f(x) h(\phi_{0,t}(x,\omega)) J_{0,t}(x,\omega) dx$$

by the formula of the change of variables.

Remark 4.7. Consider the case of constant diffusion coefficient $\sigma(x) \equiv \sigma$ and assume that the drift possessed the property $div g = 0$. Then (4.8) has the form

$$\langle S_{t,0}^g(\omega), h \rangle = \langle f, h \circ \phi_{0,t}^g(\omega) \rangle, \quad h \in \mathcal{H}^1, \qquad (4.9)$$

since in this case $J_{0,t}^g(\omega) = Id$ is the identity map.

Consider a linear PDE

$$\frac{df}{dt} = L^g f - \gamma(t), \quad f(0) = f_0, \qquad (4.10)$$

where
$$L^g = (g, \nabla) + L_0,$$

$$L_0 f = \frac{1}{2} F_{ij} \nabla_i \nabla_j f + m_j \nabla_j f,$$

and
$$F_{ij} = \sigma_{ik} \sigma_{jk}, \quad m_j = \nabla_j \sigma_{ik} \sigma_{jk},$$

and $g \in C^1$ is a given bounded smooth function and $f_0 \in C^1$ (or more generally $f_0 \in \mathcal{D}'$).

To construct a probabilistic representation of a weak solution to (4.8) in the case when the initial data f_0 is a C^1 function (or even a distribution $f_0 \in \mathcal{D}'$) we consider the stochastic process

$$\lambda(t) = f_0 - \int_0^t \gamma(\tau) \circ \phi_{0,\tau}^g d\tau \tag{4.11}$$

and define its composition with a stochastic flow $\psi_{t,0}^g(x)$ solving (4.7). Recall that $\psi_{t,0}^g(x)$ is inverse to the stochastic flow $\phi_{0,t}^g(x)$ generated by the solution $\xi(t)$ of (4.2).

It is proved in [19] that the generalized solution of (4.8) is given by the generalized expectation of $\lambda(t) \circ \psi_{t,0}^g(x)$.

To define the generalized expectation we consider the Sobolev spaces $W^{k,q}$ or the weighted Sobolev spaces $\mathcal{S}_{k,q}$ defined in section 1 and check that $E\lambda(t) \circ \psi_{t,0}^g(x)$ is well defined.

Lemma 4.8. *For each integer k and $q > 1$, $T > 0$ there are exist positive constants $c_{k,,q,T}, c'_{k,,q,T}$ depending only on the flow $\psi_{t,0}$ such that for any $t \in [0,T]$*

$$E\|\lambda(t) \circ \psi_{t,0}\|_k^q \le c_{k,,q,T}\|f_0\|_{k,q}^q + c'_{k,,q,T} \int_0^t \|\gamma(\tau)\|_{k,q}^q d\tau, \tag{4.12}$$

for all $f_0 \in W^{k,q}$ and $\gamma(t) \in W^{k,q}$.

If $f_0 \in W^{k,q}$ then by this lemma for any $h \in W^{-k,q}$ there exists

$$\langle S_{0,t}, h \rangle = E\langle \lambda(t) \circ \psi_{t,0}, h \rangle,$$

and $S_{0,t}$ can be considered as an element from $W^{k,q}$. This element will be called the generalized expectation of $\lambda(t) \circ \psi_{t,0}$ and denoted by $E[\lambda(t) \circ \psi_{t,0}]$.

For $k = 1, q = 2$ we consider $\langle \bar{S}_{t,0}^g, h \rangle = E\langle f \circ \psi_{t,0}^g, h \rangle$ which is a continuous linear functional on $\mathcal{H}^1$ and can be regarded as an element of $\mathcal{H}^{-1}$. Set

$$U_{t,0} f = E[f \circ \psi_{t,0}] \tag{4.13}$$

and call it the generalized expectation of $f \circ \psi_{t,0}$. It is easy to see that $U_{t,0}$ is a linear map form $\mathcal{H}^{-1}$ into itself. Moreover the family $U_{t,s} f = E[f \circ \psi_{t,s}]$ possesses the evolution property $U_{t,\tau} U_{\tau,s} = U_{t,s}$ for any $0 \le s \le t \le T$. It can be immediately deduced from the evolution properties of $\phi_{s,t}^g$ and $J_{s,t}^g$.

Finally we compute the infinitesimal operator of the evolution family $U_{t,s}$. To this end we need a version of the Itô formula.

Theorem 4.9.(The generalized Itô formula) *Let $f(t) \in \mathcal{H}^1$ be a continuous in t nonrandom function. Then, given stochastic flows $\phi_{0,t}, \psi_{t,0}$ generated by (4.1), (4.6) the following relations hold*

$$f(t) \circ \phi_{0,t}^g = f(0) + \int_0^t [\frac{\partial f(\theta)}{\partial \theta} \circ \phi_{0,\theta}^g + L_0 f(\theta) \circ \phi_{0,\theta}^g]d\theta +$$

$$\int_0^t \nabla_i f(\theta) \circ \phi_{0,\theta}^g d[\phi^g]_{0,\theta}^i$$

and

$$f(t) \circ \psi_{t,0}^g = f(0) + \int_0^t [\frac{\partial}{\partial \theta}[f(\theta)] \circ \psi_{\theta,0}^g + L_0[f(\theta) \circ \psi_{\theta,0}^g]]d\theta + \quad (4.14)$$

$$\int_0^t \nabla_i[f(\theta) \circ \psi_{\theta,0}^g]\sigma dw + \int_0^t \nabla_i[f(\theta) \circ \psi_{\theta,0}^g]g(\theta)d\theta.$$

Here we understand the action of the operator L_0 in the sense of generalized functions.

The proof of theorem 4.9 employs the classical Itô formula for C^2- smooth functions f_ε that approximate the C^1 function f, uses equations (4.2) and (4.6) for the flows $\phi_{0,t}$ and $\psi_{t,0}$, respectively, and then justifies the passage to the limit under the integral sign in the integral identity. The details can be found in [19] for a much more general case.

Let us come back to the parabolic equation (4.10) and set $\gamma = 0$. We can show that the stochastic flow $\psi_{t,0}$ gives rise to an evolution family acting in spaces of of distributions and the function $f(t) = E[f_0 \circ \psi_{t,0}]$ is a weak solution of (4.10) with $\gamma = 0$.

Theorem 4.10. *Assume that the coefficients of the stochastic equation (4.2) satisfy **C 4.1** and $\psi_{t,0}^g$ is generated by the solution of (4.6). Then, for any functions $f_0, g(t) \in \mathcal{H}^1$ the relation*

$$f(t) = E[f_0 \circ \psi_{t,0}^g]$$

defines the unique generalized solution to the problem (4.10) with $\gamma = 0$. The restriction of $U_{t,0}^g$ to $\mathcal{H}^1$ defines a strongly continuous family of evolution mappings acting on the space $\mathcal{H}^1$. The domain of definition of its infinitesimal operator $\mathcal{A}^g$ (in a weak sense) contains the subspace $\mathcal{H}^1$ and $\mathcal{A}^g f = L^g f$ for any $u \in \mathcal{H}^1$.

Proof. From the relation (4.13) and the properties of stochastic flows we deduce that the relation

$$\langle U^g(t)f_0, h \rangle = \langle E[f_0 \circ \psi_{t,0}^g], h \rangle$$

defines a continuous linear functional on $\mathcal{H}^1$. Thus, we can treat $U^g(t)f_0$ as an element from $\mathcal{H}^1$. It follows from the representation $U_{t,0}^g f_0(x) = E[f_0 \circ \psi_{t,0}^g(x)]$ that $U^g(t)$ is a linear mapping from the space $\mathcal{H}^1$ into itself. Note that the above definition of the family $U_{t,s}^g$ through the integral identity allows to check that it possesses the evolution property

$$\langle U_{t,\tau}^g U_{\tau,s}^g f_0, h \rangle = \langle U_{t,s}^g f_0, h \rangle.$$

Indeed, by the Markov property of the process $\psi_{t,0}^g(x)$ we deduce that

$$\langle U_{t,s}^g f_0, h \rangle = \langle U_{t,\tau}^g U_{\tau,s}^g f_0, h \rangle = E[\langle U_{\tau,s}^g f_0, h \circ \phi_{\tau,t}^g J_{\tau,t}^g \rangle] =$$

$$E[\langle f_0, [v \circ \phi_{s,\tau}^g] J_{s,\tau}^g \rangle|_{v=h\circ\phi_{\tau,t}^g J_{\tau,t}^g}] = E\langle f_0, [h \circ \phi_{s,\tau}^g \circ \phi_{\tau,t}^g] J_{s,\tau}^g] \circ \phi_{\tau,s}^g J_{\tau,t}^g \rangle =$$

$$E[\langle f_0, [h \circ \phi_{s,t}^g] J_{s,t}^g \rangle] = \langle U_{s,t}^g f, h \rangle.$$

Now we apply the generalized Itô formula to obtain the relation

$$E[f_0 \circ \psi_{t,0}^g] = f_0 + E[\int_0^t L^g(f_0 \circ \psi_{\theta,0}^g)d\theta].$$

Note that in the latter expression each summand belongs to $\mathcal{H}^1$. In addition,

$$E[\int_0^t \langle L^g(f_0 \circ \psi_{\theta,0}^g), h \rangle d\theta] = \int_0^t \langle E[f_0 \circ \psi_{\theta,0}^g], (L^g)^*h \rangle d\theta =$$

$$\int_0^t \langle L^g(E[f_0 \circ \psi_{\theta,0}^g]), h \rangle d\theta,$$

that yields

$$E[f_0 \circ \psi_{t,0}^g] = f_0 + \int_0^t L^g(E[f_0 \circ \psi_{\theta,0}^g])d\theta.$$

In other words

$$U_{t,0}^g f_0 = f_0 + \int_0^t L^g U_{\theta,0}^g f_0 d\theta.$$

As the result we get that $f(t) = E[f_0 \circ \psi_{t,0}]$ satisfies (4.8) and $f(0) = f_0$.

One can prove the corresponding result in the case $\gamma(t) \neq 0$ in a similar way applying the above reasons to $\lambda(t)$ of the form (4.11) instead of f_0.

Theorem 4.11. *Given tempered distributions f_0 and $\gamma(t)$ define $\lambda(t)$ by (4.11). Then $U(t) = E[\lambda(t) \circ \psi_{t,0}^g]$ defines the unique solution of equation (4.10) if $\psi_{t,0}^g$ satisfies (4.6) and $\phi_{0,t}$ is its inverse.*

Proof. By the generalized Ito formula we get

$$\lambda(t) \circ \psi_{t,0}^g = f_0 - \int_0^t \gamma(\tau)d\tau + \int_0^t \nabla_i(\lambda(\tau) \circ \psi_{\tau,0}^g)g(\tau)d\tau +$$

$$\int_0^t \nabla_i(\lambda(\tau) \circ \psi_{\tau,0}^g)\sigma(\tau)dw(\tau) + \int_0^t L_0^g(\lambda(\tau) \circ \psi_{\tau,0}^g)d\tau.$$

As a consequence we get

$$\lambda(t) \circ \psi_{t,0}^g = f_0 + \int_0^t \nabla_i(\lambda(\tau) \circ \psi_{\tau,0}^g)\sigma dw(\tau) +$$

$$\int_0^t L^g(\lambda(\tau) \circ \psi_{\tau,0}^g)d\tau - \int_0^t \gamma(\tau)d\tau. \qquad (4.15)$$

Each term in (4.15) has a generalized expectation as an element of $\mathcal{S}'$. The generalized expectation of the second term in the right hand side of (4.15) is equal to zero. For the third term we have

$$\langle E\left[\int_0^t L^g(\lambda(\tau) \circ \psi_{\tau,0}^g)d\tau\right], h\rangle = E\left[\int_0^t \langle \lambda(\tau) \circ \psi_{\tau,0}^g, [L^g]^*h\rangle d\tau\right] =$$

$$\int_0^t \langle E[\lambda(\tau) \circ \psi_{\tau,0}^g], [L^g]^*h\rangle d\tau = \int_0^t \langle L^g(E[\lambda(\tau) \circ \psi_{\tau,0}^g]), h\rangle d\tau. \quad (4.16)$$

Hence

$$E[\lambda(t) \circ \psi_{t,0}^g] = f_0 + \int_0^t L^g(E[\lambda(\tau) \circ \psi_{\tau,0}^g])d\tau - \int_0^t \gamma(\tau)d\tau. \quad (4.17)$$

Differentiating each term with respect to t we check that $U(t) = E[\lambda(t) \circ \psi_{t,0}^g]$ satisfies (4.10). In addition $\lim_{t\to 0}\langle U^g(t), h\rangle = \langle f_0, h\rangle$, that is $\lim_{t\to 0}\langle U^g(t) = f_0$ and we proved that $U^g(t)$ solves the Cauchy problem (4.10).

To prove the uniqueness of the solution to (4.8) suppose to the contrary that there exist two solutions $f(t)$ and $\tilde{f}(t)$ to (4.10). Then the function $u(t) = f(t) - \tilde{f}(t)$ satisfies $\frac{du(t)}{dt} = L^g u(t)$ and $\lim_{t\to 0} u(t) = 0$. Fix t and choose a function $h(t, \cdot) \in \mathcal{D}$. Then there exists a solution $h(\tau, x)$, $0 \le s \le \tau \le t, x \in R^3$, to the Cauchy problem

$$\frac{\partial h(\tau, x)}{\partial \tau} + [L^g]^*h(\tau, x) = 0, \quad \lim_{\tau \to t} h(\tau, x) = h(t, x).$$

If the coefficients a^g and σ^g are C^1-smooth , then there exists a unique classical solution to this Cauchy problem. As a result,

$$\langle u(t), h\rangle = \int_0^t \langle \frac{d}{d\theta}u(\theta), h(\theta)\rangle d\theta + \int_0^t \langle u(\theta), \frac{d}{d\theta}h(\theta)\rangle d\theta =$$

$$\int_0^t \langle L^g u(\theta), h(\theta)\rangle d\theta - \int_0^t \langle u(\theta), [L^g]^*h(\theta)\rangle d\theta = 0.$$

5 A probabilistic approach to the Navier-Stokes system

Let us come back to the Navier-Stokes system

$$\frac{\partial u}{\partial t} + (u, \nabla)u = \frac{\sigma^2}{2}\Delta u - \nabla p, \quad u(0, x) = u_0(x), \quad x \in R^3 \quad (5.1)$$

$$-\Delta p = \gamma, \quad (5.2)$$

with γ defined by (1.4).

Our main purpose in this section is to construct a diffusion process that allows us to obtain a probabilistic representation of a weak solution to (5.1), (5.2). To be more precise we intend to reduce the solution of this system to solution of a certain stochastic problem, to

solve it and then to verify that in this way we have constructed a weak solution of (5.1), (5.2).

Let as above $w(t), B(t)$ be standard R^3-valued independent Wiener processes defined on a probability space $(\Omega, \mathcal{F}, P)$. Given a bounded measurable function $f(x)$ and a stochastic process $\xi(t)$ we denote $E_{s,x} f(\xi(t)) \equiv E f(\xi_{s,x}(t))$ a conditional expectation under the condition $\xi(s) = x$.

In section 2 we recalled the probabilistic approach developed in our previous paper [7] that allows to construct a probabilistic representation of a C^2-smooth (classical) solution to (5.1)-(5.2) via the solution of the stochastic problem

$$d\xi(\tau) = -u(t - \tau, \xi(\tau))d\tau + \sigma dw(\tau), \tag{5.3}$$

$$u(t, x) = E_{0,x}[u_0(\xi(t)) + \int_0^t \nabla p(t - \tau, \xi(\tau))d\tau] \tag{5.4}$$

$$-2p(t, x) = E[\int_0^\infty \gamma(t, x + B(t))dt] = E[\int_0^\infty \mathrm{tr}[\nabla u]^2(t, x + B(t))dt]. \tag{5.5}$$

In this section we consider a similar stochastic system but now we choose to invert the time direction of the stochastic process itself rather then of the function u to obtain the possibility to reduce a construction of a generalized solution to the Navier-Stokes system to the construction of a solution of a stochastic problem.

Our considerations will be based on the result of sections 3 and 4. Note that since we consider the case where the diffusion coefficient σ is constant the Ito form and the Stratonovich form of a stochastic equation coincide.

Let as above $w(t), B(t)$ be standard R^3-valued independent Wiener processes defined on a probability space $(\Omega, \mathcal{F}, P)$.

Let $\phi_{0,t}(y)$ be a stochastic process satisfying the stochastic equation

$$d\phi_{0,t}(y) = u(t, \phi_{0,t}(y))dt - \sigma dw(t), \quad \phi_{0,0} = y$$

and the stochastic process $\lambda(t)$ be of the form

$$\lambda(t) = u_0 - \int_0^t \nabla p(\tau, \phi_{0,\tau})d\tau. \tag{5.6}$$

Consider the system

$$d\psi_{t,\theta,}(x) = -u(\theta, \psi_{t,\theta}(x))d\theta + \sigma d\hat{w}(\theta), \quad \psi_{t,t}(x) = x, \tag{5.7}$$

$$u(t, x) = E[u_0(\psi_{t,0}(x)) - \int_0^t \nabla p(\tau, \psi_{t,\tau}(x))d\tau]. \tag{5.8}$$

$$-2\nabla p(t, x) = E[\int_0^\infty \frac{1}{\tau}\gamma(t, x + B(\tau))B(\tau)d\tau], \tag{5.9}$$

where γ is given by (1.4) and prove the existence and uniqueness of its solution.

To this end we apply the Picard principle to the solution of the stochastic system and construct a solution to (5.7)-(5.9) by the successive approximation technique.

Set
$$u^1(t,x) = u_0(x), \quad \psi_{t,0}^0(x) = x, p^1(t,x) = 0 \qquad (5.10)$$

and consider a family of stochastic processes $\psi_{t,\theta}^k(x)$ and families of vector fields $u^k(t,x)$ and scalar functions $p^k(t,x)$ given by the following relations

$$d\psi_{t,\theta}^k = -u^k(\theta, \psi_{t,\theta}^k)d\theta + \sigma d\hat{w}(\theta), \quad \psi_{t,t}^k = x, \qquad (5.11)$$

$$u^{k+1}(t,x) = E[u_0(\psi_{t,0}^k(x)) - \int_0^t \nabla p^{k+1}(\tau, \psi_{t,\tau}^k(x))d\tau], \qquad (5.12)$$

$$-2p^{k+1}(t,x) = \int_0^\infty E[\gamma^{k+1}(t, x + B(\tau))]d\tau, \qquad (5.13)$$

where
$$\gamma^{k+1}(t,x) = Tr[\nabla u^k(t,x)\nabla u^{k+1}(t,x)]. \qquad (5.14)$$

Note that for a fixed k the first stochastic equation (5.11) that determines the family of stochastic processes $\psi_{t,0}^k(x)$ may be solved independently on equations (5.12)-(5.14). Then given the process $\psi_{t,0}^k(x)$ and keeping in mind the properties of the function p^k that satisfies the Poisson equation

$$-\Delta p^{k+1}(t,x) = \gamma^{k+1}(t,x), \qquad (5.15)$$

one has to compute $\nabla p^k(t,x)$, $u^{k+1}(t,x)$ by (5.12), (5.13).

To investigate the convergence of the stochastic processes $\psi_{t,0}^k(x)$ and functions $u^k(t,x), p^k(t,x)$ defined above we need some auxiliary results concerning the behavior of solutions of stochastic equations.

Let $g \in V$ be a given function. Consider the stochastic equation

$$d\psi_{t,\theta}^g = -g(\theta) \circ \psi_{t,\theta}^g d\theta + \sigma d\hat{w}(\theta), \quad \psi_{t,t}^g(x) = x \qquad (5.16)$$

and define vector fields $u^g(t,x)$ and $\nabla p^g(t,x)$ by

$$u^g(t,x) = E[u_0(\psi_{t,0}^g(x)) - \int_0^t \nabla p^g(\tau, \psi_{t,\tau}^g(x))d\tau], \qquad (5.17)$$

$$-2p^g(t,x) = \int_0^\infty E[\gamma^g(t, x + B(\tau))]d\tau, \qquad (5.18)$$

$$\gamma^g(t,x) = Tr[\nabla g \nabla u^g](t,x). \qquad (5.19)$$

Recall that p^g solves the Poisson equation

$$-\Delta p^g = Tr[\nabla g \nabla u^g]. \qquad (5.20)$$

To investigate the convergence of the stochastic processes $\psi_{t,0}^k(x)$ and functions $u^k(t,x), p^k(t,x)$ defined above we need some auxiliary results concerning the behavior of solutions of stochastic equations. Moreover

along with the system (5.16) – (5.18) we will need the system to describe the process $\eta^k(\tau) = \nabla \psi_{t,\tau}^k(x)$ and the functions $\nabla u^k(t,x)$ and $\nabla p^k(t,x)$.

To derive the necessary apriori estimates we start with the consideration of a linearized system.

Let $g(t)$ be a given vector field. Consider the stochastic equation

$$d\psi_{t,\theta}^g = -g(\theta) \circ \psi_{t,\theta}^g d\theta + \sigma d\hat{w}(\theta), \quad \psi_{t,t}^g(x) = x \tag{5.21}$$

and define the vector fields $u^g(t,x)$ and $\nabla p^g(t,x)$ by

$$u^g(t,x) = E[u_0(\psi_{t,0}^g(x)) - \int_0^t \nabla p^g(\tau, \psi_{t,\tau}^g(x))d\tau], \tag{5.22}$$

$$-2p^g(t,x) = \int_0^\infty E[\gamma^g(t, x + B(\tau))]d\tau, \tag{5.23}$$

$$\gamma^g(t,x) = Tr[\nabla g \nabla u^g](t,x). \tag{5.24}$$

Finally we derive from (5.23) the relation

$$-2\nabla p^g(t,x) = \int_0^\infty E[\frac{1}{\tau}\gamma^g(t, x + B(\tau))B(\tau)]d\tau \tag{5.25}$$

by applying the Bismut – Elworthy – Li formula first checking the conditions that validate such an application are satisfied. Below we will need some estimates of a solution of the Poisson equation from section 3. For convenience of references we formulate them in the following statement.

Lemma 5.1.

1. Let $\gamma^g \in L^q(R^3) \cap L^m(R^3)$ for some $1 \leq q < \frac{3}{2} < 3 < m < \infty$. Then

$$\|\nabla p^g\|_\infty \leq C_{qm}(\|\gamma^g\|_q + \|\gamma^g\|_m)$$

$$\|\nabla_i \nabla_j p^g\|_\infty \leq C(\|\gamma^g\|_q + [\gamma^g]_\alpha).$$

2. Let $\gamma^g \in L^r(R^3)$ for $1 < r < \infty$. Then $p^g \in W_0^{2,r}(R^3)$ and the Calderon- Zygmund inequality

$$\|\nabla_i \nabla_j p^g\|_{r,loc} \leq C_1 \|\gamma^g\|_{r,loc}$$

holds.

Let Lip be the subspace of the space $C(R^1 \times R^3, R^3)$ of continuous (in $t \in [0,T], x \in R^3$), bounded functions which consists of Lipschitz-continuous (in x) functions g such that

$$\|g(t,x) - g(t,y)\| \leq L_g(t)\|x - y\|, \quad t \in [0,T] \quad x,y \in R^3,$$

where $\|\cdot\|$ is the norm in R^3.

Condition **C 5.1**

Let $g(t,x) \in R^3$ be a vector field defined on $[0,T] \times R^3$ that belongs to $C^{1,\alpha}(R^3, R^3), 0 < \alpha \leq 1$ for a fixed $t \in [0,T]$ and satisfies the following estimates:

1. $\|g(t)\|_{L^q_{loc}} \le N_g(t)$ for some q to be specified below, $\|g(t)\|_\infty \le K_g(t)$ and

$$\|g(t,x)-g(t,y)\| \le L_g(t)\|x-y\|, \quad \|\nabla g(t,x)-\nabla g(t,y)\| \le L_g^1(t)\|x-y\|.$$

2. $\|\nabla g(t)\|_\infty \le K_g^1(t), \quad \|\nabla g(t)\|_{r,loc} \le N_g^1(t)$, where $K_g(t), L_g(t)$, $N_g(t)$ and $K_g^1(t), L_g^1(t), N_g^1(t)$ are positive functions bounded on an interval $[0,T]$, $r=m$ and $r=q$ for $1 < q < \frac{3}{2} < 3 < m < \infty$.

Set $\psi(\tau) = \psi_{t,\tau}(x)$ and consider the stochastic equation

$$\psi(\tau) = x - \int_\tau^t g(\tau_1, \psi(\tau_1))d\tau_1 + \int_\tau^t \sigma d\hat{w}(\tau_1), \qquad (5.26)$$

with $0 \le \tau \le t < T$ for a certain constant T. If we are interested in the particular dependence of the process $\psi(\tau)$ on the parameters t, x and g, we write $\psi(\tau) = \psi_{t,x}^g(\tau)$.

Lemma 5.2. *Assume that* **C 5.1** *holds. Then there exists a unique solution $\psi_x^g(\tau)$ of (5.21) that satisfies the following estimates:*

$$E\|\psi_x^g(\tau)\|^2 \le 3[\|x\|^2 + \sigma^2(t-\tau) + (t-\tau)\int_\tau^t [K_g^2(\tau_1)]d\tau_1], \quad (5.27)$$

$$E\|\psi_x^g(\tau) - \psi_y^g(\tau)\| \le \|x-y\|e^{\int_\tau^t L_g(\theta)d\theta}, \qquad (5.28)$$

$$E\|\psi_x^g(\tau) - \psi_x^{g_1}(\tau)\| \le \int_\tau^t \|g(\tau_1) - g_1(\tau_1)\|_\infty d\tau_1 e^{\int_\tau^t L_g(\theta)d\theta}. \qquad (5.29)$$

Proof. The proof of the estimates of this lemma is standard. We only show the proof of (5.28). In view of **C 5.1** we have

$$E\|\psi_x^g(\tau) - \psi_y^g(\tau)\| \le \|x-y\| + \int_\tau^t L_g(\tau_1)\|\psi_x^g(\tau_1) - \psi_y^g(\tau_1)\|d\tau_1,$$

where $0 \le \tau \le t \le T$ with some constant T to be chosen later. Finally, by Gronwall's lemma, we get

$$E\|\psi_x^g(\tau) - \psi_y^g(\tau)\| \le \|x-y\|e^{\int_\tau^t L_g(\theta)d\theta}.$$

Along with the equations (5.21)-(5.23) we will need below the equations for the mean square derivative $\eta(t) = \nabla\psi_{t,0}(x)$ of the diffusion process $\psi_{t,0}(x)$ that satisfies (5.21) and the gradient $v(t,x) = \nabla u(t,x)$ of the function $u(t,x)$ of the form (5.22).

Lemma 5.3 *Assume that* **C 5.1** *holds. Then the process $\eta^g(\tau) = \nabla\psi_{t,\tau}^g$ satisfies the stochastic equation*

$$d\eta^g(\tau) = -\nabla g(\tau, \psi_{t,\tau}^g)\eta^g(\tau)d\tau, \quad \eta^g(t) = I, \qquad (5.30)$$

where I is the identity map. Furthermore the process $\eta^g(\tau)$ possesses the following properties.

The determinant $\det\eta(\tau)$ is equal to 1, i. e.

$$\det\eta^g(\tau) = J_{t,\tau} = 1,$$

and the estimate

$$\|\eta^g(\tau)\| \le e^{\int_\tau^t K_g^1(\theta)d\theta} \qquad (5.31)$$

holds.

In addition the following integration by parts formula is valid

$$\int_{R^3} f(\psi_x^g(\tau))dx = \int_{R^3} f(x)dx, \quad f \in L^1(R^3). \qquad (5.32)$$

Proof. Under the above assumptions the first statement immediately follows from the results of the stochastic differential equation theory. By a direct computation one can check that $J_{t,\tau}$ satisfies the linear equation

$$dJ_{t,\tau} = -div\, g(\psi_{t,\tau}^g)J_{t,\tau}d\tau, \quad J_{t,t} = I$$

and since $div\, g = 0$ we get the second statement that yields the integration by parts formula (5.32). Finally (5.31) is deduced from the inequality

$$E\|\eta(\tau)\| \leq 1 + \int_\tau^t K_g^1(\theta)E\|\eta(\theta)\|d\theta$$

by the Gronwall lemma.

In the sequel we denote by $\eta^{x,g}(t)$ the solution of the equation

$$d\eta^{x,g}(\tau) = \nabla g(t, \psi_{t,\tau}(x))\eta^{x,g}(\tau)d\tau, \quad \eta^{x,g}(0) = I$$

if we will be interested in the properties of the process $\eta^{x,g}(t)$. One can easily check that

$$\|\eta^{x,g}(\tau) - \eta^{y,g}(\tau)\| \leq \int_\tau^t \|\nabla g(\theta, \psi_{t,\theta}(x)) - \nabla g(\theta, \psi_{t,\theta}(y))\|d\theta e^{\int_\tau^t K_g^1(\theta)d\theta}$$

$$\leq \int_\tau^t L_g^1(\theta)\|\psi_{t,\theta}(x)) - \psi_{t,\theta}(y))\|d\theta$$

and by (5.28) we have

$$\|\eta^{x,g}(\tau) - \eta^{y,g}(\tau)\| \leq C(\tau)\|x - y\|$$

where $C(\tau)$ is a bounded function over a certain interval $[0, T_1]$ depending on g.

Let us state conditions on initial data u_0 of the N-S system.

We say that **C 5.2** holds if for $0 < \alpha \leq 1$ the initial vector field $u_0 \in C^{1,\alpha}$ satisfies the following estimates

$$\|u_0\|_\infty \leq K_0, \quad \|\nabla u_0\|_\infty \leq K_0^1, \quad \|u_0\|_{r,loc} \leq M_0, \quad \|\nabla u_0\|_{r,loc} \leq M_0^1$$

with r to be specified below and let L_0, L_0^1 be Lipschitz constants for the functions u_0 and ∇u_0 respectively.

Lemma 5.4. Assume that $g(t, x)$ satisfies **C 5.1** and u_0 satisfies **C 5.2** with $r = q$ and $r = m$ for $1 < q < \frac{3}{2} < 3 < m < \infty$. Then the vector field $u^g(t, x)$ given by

$$u^g(t, x) = E[u_0(\psi_{t,x}^g(0)) - \int_0^t \nabla p^g(\tau, \psi_{t,\tau}^g(x))d\tau] \qquad (5.33)$$

satisfies the following estimate

$$\|u^g(t)\|_\infty \le K_0 + \int_0^t C_{qm}[\|\nabla g(\tau)\nabla u^g(\tau)\|_{q,loc} + \|\nabla g(\tau)\nabla u^g(\tau)\|_{m,loc}]d\tau. \tag{5.34}$$

The proof of the estimate can easily be obtained by direct computation from (5.33) using the estimates of the Newton potential given in lemma 5.1.

Lemma 5.5. *Assume that conditions of lemma 5.4 hold. Then given the function $u^g(t,x)$ of the form (5.33) the function $\nabla u^g(t,x)$ admits a representation of the form*

$$\nabla u^g(t,x) = E[\nabla u_0(\psi_{t,0}^g(x))\eta^{x,g}(t) -$$

$$\int_0^t \frac{1}{\sigma(t-\tau)} \nabla p^g(\tau, \psi_{t,\tau}^g(x)) \int_\tau^t \eta^{x,g}(\theta) d\hat{w}(\theta)d\tau] \tag{5.35}$$

and the estimate

$$\|\nabla u^g(t)\|_\infty \le e^{\int_0^t K_g^1(\theta)d\theta} K_0^1 +$$

$$\int_0^t C_{qm} \frac{1}{\sigma\sqrt{t-\tau}} e^{\int_\tau^t K_g^1(\theta)d\theta} K_g^1(\tau)[\|\nabla u^g(\tau)\|_{q,loc} + \|\nabla u^g(\tau)\|_{m,loc}]d\tau \tag{5.36}$$

holds for $1 < q < \frac{3}{2} < 3 < m < \infty$.

Proof. To derive (5.35) we compute directly the gradient of the first term in (5.34) and apply the Bismut-Elworthy -Li formula [8] to compute the gradient of the second term in this relation. To verify the estimate (5.36) we use the above estimates for the process $\eta(t)$ and the estimates of the Newton potential derivative from lemma 5.1. Then we obtain

$$\|\nabla u^g(t)\|_\infty \le e^{\int_0^t K_g^1(\theta)d\theta} K_0^1 +$$

$$\int_0^t C_{m,q} \frac{1}{\sigma\sqrt{t-\tau}} e^{\int_\tau^t K_g^1(\theta)d\theta} [\|\nabla g(\tau)\nabla u^g(\tau)\|_{m,loc} + \tag{5.37}$$

$$\|\nabla g(\tau)\nabla u^g(\tau)\|_{q,loc}]d\tau] \le e^{\int_0^t K_g^1(\theta)d\theta} K_0^1 +$$

$$\int_0^t C_{qm} \frac{1}{\sigma\sqrt{t-\tau}} e^{\int_\tau^t K_g^1(\theta)d\theta} K_g^1(\tau)[\|\nabla u^g(\tau)\|_{q,loc} + \|\nabla u^g(\tau)\|_{m,loc}]d\tau].$$

Now we have to derive the estimate for the function $\|\nabla u(t)\|_{r,loc}$.

Lemma 5.6. *Assume that the conditions of lemma 2.4 hold. Then for $1 < r < \infty$ the function $u^g(t,x)$ given by (5.33) satisfies the estimate*

$$\|\nabla u^g(t)\|_{r,loc} \le e^{\int_0^t K_g^1(\theta)d\theta} \|\nabla u_0\|_{r,loc} + \tag{5.38}$$

$$C_{qm} \int_0^t e^{\int_0^\tau K_g^1(\theta)d\theta} K_g^1(\tau)\|\nabla u^g(\tau)\|_{r,loc}, d\tau$$

with a constant C depending on r and a certain constant T to be specified later.

Proof. Recall that along with (5.35) $\nabla u^g(t,x)$ admits the representation

$$\nabla u^g(t,x) = E[\nabla u_0(\psi^g_{t,0}(x))\eta^{x,g}(t) - \int_0^t \nabla^2 p^g(\tau, \psi^g_{t,\tau}(x))\eta^{x,g}(\tau)d\tau].$$

To derive the estimate for $\|\nabla u(t)\|^r_{r,loc} = \int_G \|\nabla u(t,x)\|^r dx$ (where G is an arbitrary compact in R^3) by the triangle inequality we get

$$\|\nabla u^g(t)\|_{r,loc} \le \alpha_1 + \alpha_2,$$

where

$$\alpha_1 = \left(\int_G E[\|\nabla u_0(\psi^g_{t,0}(x))\eta^{x,g}(t)\|^r]dx\right)^{\frac{1}{r}},$$

$$\alpha_2 = \left(\int_G \int_0^t \|\nabla^2 p^g(\tau, \psi^g_{t,\tau}(x)))\eta^{x,g}(\tau)\|^r d\tau dx\right)^{\frac{1}{r}}.$$

To estimate α_1 we apply the Hölder inequality and recall that $\psi_{t,\tau}(x)$ preserves the volume. As a result we have

$$\alpha_1 \le \left(\int_G (E[\|\nabla u_0(\psi^g_{t,0}(x))\|^2]E[\|\eta^g(t)\|^2])^{\frac{r}{2}}dx\right)^{\frac{1}{r}} \le$$

$$\|\nabla u_0\|_{r,loc}e^{\int_0^t K^1_g(\theta)d\theta}.$$

To estimate α_2 we apply the Calderon-Zygmund inequality and the above property of $\psi_{t,\tau}(x)$ to obtain

$$\alpha_2^r \le C_r \int_0^t e^{\int_0^\tau K^1_g(\theta)d\theta} K^1_g(\tau) \int_G \|\nabla u(\tau,x)\|^r dx d\tau.$$

Combining the above estimates for α_1 and α_2 we obtain the required estimate

$$\|\nabla u^g(t)\|_{r,loc} \le e^{\int_0^t K^1_g(\theta)d\theta}[\|\nabla u_0\|_{r,loc}+$$

$$C_r \int_0^t e^{\int_0^\tau K^1_g(\theta)d\theta} K^1_g(\tau)\|\nabla u^g(\tau)\|_{r,loc}d\tau].$$

Theorem 5.7. *Assume that conditions* **C 5.1** *and* **C 5.2** *hold. Then there exists an interval* $\Delta_1 = [0,T_1]$ *and functions* $\alpha(t)$, $\beta(t)$, κ *bounded for* $t \in \Delta_1$, *such that, if for all* $t \in \Delta_1$, $\|g(t)\|_\infty \le \kappa(t)$ *and* $\|\nabla g(t)\|_\infty \le \alpha(t)$, $\|\nabla g(t)\|_r \le \beta_r(t)$ *then the function* $\|\nabla u^g(t,x)\|$ *(where* $u^g(t,x)$ *is given by (5.21)) satisfies the estimates*

$$\|u^g(t)\|_\infty \le \kappa(t), \quad \|\nabla u^g(t)\|^2_\infty \le \alpha(t), \quad \|\nabla u^g(t)\|^2_{r,loc} \le \beta_r(t) \tag{5.39}$$

for $r = q$ *and* $r = m$ *and* $1 < m < \frac{3}{2} < 3 < q < \infty$.

Proof. Analyzing the above estimates for the functions $u^g(t,x)$ and $\nabla u^g(t,x)$ we get the following estimates

$$\|\nabla u^g(t)\|_\infty \le e^{\int_0^t K^1_g(\theta)d\theta} K^1_0+ \tag{5.40}$$

$$\int_0^t C_{qm} e^{\int_0^\tau K^1_g(\theta)d\theta} K^1_g(\tau)[\|\nabla u^g(\tau)\|_{q,loc} + \|\nabla u^g(\tau)\|_{m,loc}]d\tau],$$

$$\|\nabla u(t)\|_{r,loc} \leq e^{\int_0^t K_g^1(\theta)d\theta}[\|\nabla u_0\|_{r,loc}+ \tag{5.41}$$

$$C_r \int_0^t e^{\int_0^\tau K_g^1(\theta)d\theta} K_g^1(\tau)\|\nabla u^g(\tau)\|_{r,loc}d\tau].$$

To derive the required estimates consider the integral equations

$$\alpha(s) = e^{\int_s^t \alpha(\theta)d\theta} K_0^1 + C_{qm} \int_s^t e^{\int_s^\tau \alpha(\theta)d\theta} \alpha(\tau)[n_q(\tau) + n_m(\tau)]d\tau, \tag{5.42}$$

$$n_r(s) = e^{\int_s^t \alpha(\theta)d\theta}\|\nabla u_0\|_r + C_r \int_s^t e^{\int_s^\tau \alpha(\theta)d\theta} n_r(\tau)\alpha(\tau)d\tau$$

for $r = q$ and $r = m$ and $C_{qm}^1 = max(C_q, C_m)$. Finally we consider the equation

$$\beta(s) = e^{\int_s^t \alpha(\theta)d\theta}\beta_0 + C_{qm}^1 \int_s^t e^{\int_s^\tau \alpha(\theta)d\theta}\alpha(\tau)\beta(\tau)d\tau, \tag{5.43}$$

where $\beta(\tau) = n_q(\tau) + n_m(\tau)$, and

$$\|\nabla u_0\|_{q,loc} + \|\nabla u_0\|_{m,loc} = n_q(0) + n_m(0) = \beta_0.$$

Next instead of the above system of integral equations we consider the system of ODEs

$$\frac{d\alpha}{ds} = -\alpha^2(s) - C_{qm}\alpha(s)\beta(s), \quad \alpha(t) = K_0^1, \tag{5.44}$$

$$\frac{d\beta}{ds} = -\alpha(s)\beta(s) - C_{qm}^1\alpha(s)\beta(s), \quad \beta(t) = \beta_0. \tag{5.45}$$

By classical results of the ODE theory we know that there exists an interval $[0, T_1]$ depending on K_0^1, N_0^1 and C, C_{qm} such that the system (5.44), (5.45) has a bounded solution defined on this interval.

To prove the convergence of functions $u^k(t, x), \nabla u^k(t, x)$ we need one more auxiliary estimate. Actually, we have proved that $u^k(t) \in Lip$ with the Lipschitz constant independent of k. It remains to prove that $\nabla u^k(t)$ have the same property.

Lemma 5.8. *Assume that* **C 5.1** *and* **C 5.2** *hold. Then the function* $\nabla u^g(t)$ *satisfies the estimate*

$$\|\nabla u^g(t, x) - \nabla u^g(t, y)\| \leq N_1^g(t)\|x - y\|^\alpha \; \text{ if } t \in [0, T_1]$$

for any $x, y \in G$ *where* G *is a compact in* R^3 *and the positive function* $N_1^g(t)$ *depending on parameters in conditions* **C 5.1** *and* **C 5.2** *is bounded over the interval* $[0, T_1]$ *defined in theorem 5.7.*

Proof. Applying the integration by parts Bismut – Elworthy – Li formula to (5.33) we deduce the following expression for the gradient of the function $u(t, x)$

$$\nabla u^g(t, x) = E[\nabla u_0(\psi_{t,0}^g(x))\eta^{x,g}(t)- \tag{5.46}$$

$$\int_0^t \frac{1}{\sigma(t-\tau)} \nabla p^g(\tau, \psi_{t,\tau}^g(x)) \int_\tau^t \eta^{x,g}(\theta) d\hat{w}(\theta) d\tau].$$

It results from (5.46) that

$$\|\nabla u^g(t,x) - \nabla u^g(t,y)\| \leq \kappa_1 + \kappa_2 + \kappa_3 + \kappa_4,$$

where

$$\kappa_1 = E[\|\nabla u_0(\psi_{t,0}^g(x)) - \nabla u_0(\psi_{t,0}^g(y))\| \|\eta^{x,g}(t)\|],$$

$$\kappa_2 = E[\|\nabla u_0(\psi_{t,0}^g(y))\| \|\eta^{x,g}(0) - \eta^{y,g}(0)\|],$$

$$\kappa_3 = \int_0^t \frac{1}{\sigma(t-\tau)} E[\|\nabla p^g(\tau, \psi_{t,\tau}^g(x)) - \nabla p^g(\tau, \psi_{t,\tau}^g(y))\| \| \int_\tau^t \eta^{x,g}(\theta) d\hat{w}(\theta)\|] d\tau,$$

$$\kappa_4 = \int_0^t \frac{1}{\sigma(t-\tau)} E\left[\|\nabla p^g(\tau, \psi_{t,\tau}^g(y))\| \int_\tau^t [\eta^{x,g}(\theta) - \eta^{y,g}(\theta)] d\hat{w}(\theta)\| \right] d\tau.$$

One can easily check using the estimates stated in lemmas 5.3 – 5.5 that under conditions **C 5.1**, **C 5.2**

$$\kappa_1 \leq L_0^1 E \|\psi_{t,0}^g(x) - \psi_{t,0}(y)\| e^{\int_0^t K_g^1(\theta)d\theta} \leq \|x-y\| L_0^1 e^{\int_\tau^t L_g(\theta)d\theta}$$

and

$$\kappa_2 \leq K_0^1 E \|\eta^{x,g}(t) - \eta^{y,g}(t)\| \leq \|x-y\| \int_\tau^t K_g^1(\theta) e^{\int_\theta^t K_g^1(\theta_1)d\theta_1} d\theta.$$

To derive the estimates for κ_3 and κ_4 we recall (see lemma 5.1) that the solution of the Poisson equation $-\Delta p^g = \gamma^g$ satisfies the estimates $\|\nabla_i \nabla_j p^g\|_\infty \leq C(\|\gamma^g\|_q + [\gamma^g]_{\alpha,G})$, $\|\nabla_i \nabla_j p^g\|_r \leq \|\gamma^g\|_r$ and $\|\nabla p^g\|_\infty \leq C_{qm}(\|\gamma^g\|_q + \|\gamma^g\|_m)$. Hence we obtain the inequalities

$$\kappa_3 \leq \int_0^t \frac{1}{\sigma\sqrt{t-\tau}} (E\|\psi_{t,\tau}^g(x) - \psi_{t,\tau}(y)\|^2)^{\frac{1}{2}} (\|\gamma^g(\tau)\|_q + [\gamma^g(\tau)]_{\alpha,G})$$

$$e^{\int_\tau^t K_g^1(\theta)d\theta} d\tau \leq \int_0^t \frac{1}{\sigma\sqrt{t-\tau}} (\|x-y\| L_0^1 e^{\int_\tau^t L_g(\theta)d\theta} (\|\gamma^g(\tau)\|_q +$$

$$[\gamma^g(\tau)]_{\alpha,G}) e^{\int_\tau^t K_g^1(\theta)d\theta} d\tau$$

and

$$\kappa_4 \leq \int_0^t \frac{1}{\sigma\sqrt{t-\tau}} C_{qm}(\|\gamma^g(\tau)\|_q + \|\gamma^g(\tau)\|_m)$$

$$(E\|\eta^{x,g}(\tau) - \eta^{y,g}(\tau)\|^2)^{\frac{1}{2}} d\tau \leq$$

$$\|x-y\| \int_0^t \frac{C_{qm}(\|\gamma^g(\tau)\|_q + \|\gamma^g(\tau)\|_m)}{\sigma\sqrt{t-\tau}} \int_\tau^t K_g^1(\theta) e^{\int_\theta^t K_g^1(\theta_1)d\theta_1} d\theta d\tau.$$

Denote by $\Theta(t) = sup_{x,y\in G} \frac{\|\nabla u^g(t,x) - \nabla u^g(t,y)\|}{\|x-y\|^\alpha}$ and note that

$$[\gamma^g(\tau)]_{\alpha,G} = sup_{x,y\in G}[\frac{K_g^1(\tau)\|\nabla u^g(\tau,x) - \nabla u^g(\tau,y)\|}{\|x-y\|^\alpha} +$$

$$\frac{K_u^1(\tau)\|\nabla g(\tau,x) - \nabla g(\tau,y)\|}{\|x-y\|^\alpha}] = K_g^1(\tau)\Theta(\tau) + K_u^1(\tau)[g(\tau)]_{\alpha,G}.$$

Then combining the above estimates for κ_i, $i = 1,2,3,4$, and applying the Gronwall lemma we derive the estimate

$$\Theta(t) \leq N^g(t)e^{\int_0^t K_g^1(\tau)d\tau} = N_1^g(t),$$

where $N^g(t)$ is a positive bounded function defined on the interval $[0,T_1]$ and depending on parameters in conditions **C 5.1** and **C 5.2**.

The estimates of theorem 5.7 and lemma 5.8 allow to prove the uniform convergence on compacts of the successive approximations (5.10)-(5.14) for the solutions of the system (5.7) – (5.9) in $C([0,T_1], C^{1,\alpha}(K)) \cap C([0,T_1], L^m(G) \cap L^q(G))$ for $1 < q < \frac{3}{2} < 3 < m < \infty$ and arbitrary compact G in R^3.

To this end we differentiate the system (5.10)-(5.14) and add to this system the following relations

$$d\eta_{t,\theta}^{k,x} = -\nabla u^k(\theta, \psi_{t,\theta}^k)\eta_{t,\theta}^{x,k}d\theta, \quad \eta_{t,t}^{x,k} = I, \qquad (5.47)$$

where I is the identity matrix acting in R^3 and

$$\nabla u^{k+1}(t,x) = E[\nabla u_0(\psi_{t,0}^{k+1}(x))\eta_{t,0}^{x,k} -$$

$$\int_0^t \frac{1}{\sigma(t-\tau)}\nabla p^{k+1}(\tau, \psi_{t,\tau}^k(x)) \int_\tau^t \eta_{t,\theta}^{x,k} d\hat{w}(\theta)d\tau], \qquad (5.48)$$

$$-2\nabla p^{k+1}(t,x) = \int_0^\infty \frac{1}{\tau}E[\gamma^{k+1}(t, x + B(\tau))B(\tau)]d\tau, \qquad (5.49)$$

where $\gamma^{k+1} = \nabla u^{k+1}\nabla u^k$.

Now we can prove the following assertion.

Theorem 5.9. *Assume that* **C 5.2** *holds. Then if $k \to \infty$ the functions $u^k(t), \nabla u_k(t,x)$ determined by (5.8) and (5.48) uniformly converge on compacts to a limiting function $u(t) \in C([0,T_1], C^{1,\alpha}), 0 < \alpha \leq 1$ for all $t \in [0,T_1]$, where $[0,T_1]$ is the interval such that the solution of (5.45), (5.46) is bounded on $[0,T_1]$. In addition on this interval the limiting function satisfies the estimates $\sup_x \|\nabla u(t,x)\| \leq \alpha(t)$, $\|\nabla u(t)\|_{q,loc} \leq \beta(t)$ for $1 < q < \frac{3}{2}$ where $\alpha(t)$ and $\beta(t)$ solve (5.45), (5.46).*

Proof. By theorem 5.7 we know that the mapping

$$\Phi(t,x,g) = E\left[u_0(\psi_{t,0}^g(x)) - \int_0^t \nabla p^g(\tau, \psi_{t,\tau}^g(x))d\tau\right]$$

acts in the space $C^{1,\alpha} \cap L_{q,loc} \cap L_{m,loc}$ (for a fixed $t \in [0.T_1]$) with $1 < q < \frac{3}{2} < 3 < m < \infty$.

Consider the successive approximations (5.10) –(5.14) and (5.47) – (5.49), denote by

$$S^{k+1}(t,x) = \|u^{k+1}(t,x) - u^k(t,x)\|,$$

$$n^{k+1}(t,x) = \|\nabla u^{k+1}(t,x) - \nabla u^k(t,x)\|$$

and let

$$l^k(t) = \|S^k(t)\|_\infty, \quad m_r^k(t) = \|S^k(t)\|_r,$$

$$\rho^k(t) = \|n^k(t)\|_\infty, \quad \zeta_r^k(t) = \|n^k(t)\|_r.$$

Then we obtain

$$n^{k+1}(t,x) \le L_0^1(E[\|\psi_{t,0}^k(x) - \psi_{t,0}^{k-1}(x)\|\|\eta_{t,0}^{x,k}\|]+$$

$$E[\|\psi_{t,0}^k(x)\|\|\eta_{t,0}^{x,k} - \eta_{t,0}^{x,k-1}\|]) + \int_0^t \frac{1}{\sigma(t-\tau)} E[\|\nabla p^{k+1}(\tau, \psi_{t,\tau}^k(x)) -$$

$$\nabla p^k(\tau, \psi_{t,\tau}^{k-1}(x))\|\| \int_\tau^t \eta_{t,\theta}^{x,k} d\hat{w}(\theta)\|]d\tau +$$

$$\int_0^t \frac{1}{\sigma(t-\tau)} E\left[\|\nabla p^k(\tau, \psi_{t,\tau}^k(x))\| \int_\tau^t [\eta_{t,\theta}^{x,k} - \eta_{t,\theta}^{x,k-1}]d\hat{w}(\theta)\|\right] d\tau. \quad (5.50)$$

Recall that by lemmas 5.2, 5.3 we know that

$$\sup_x E\|\psi_{t,0}^k(x) - \psi_{t,0}^{k-1}(x)\| \le \int_0^t [\|u^k(\tau) - u^{k-1}(\tau)\|_\infty]d\tau e^{\int_0^t \alpha(\tau)d\tau},$$

$$\sup_x E\|\eta_{t,0}^{x,k} - \eta_{t,0}^{x,k-1}\| \le \int_0^t \|\nabla u^k(\tau) - \nabla u^{k-1}(\tau)\|_\infty d\tau e^{\int_0^t \alpha(\tau)d\tau}$$

$$+ \sup_x \int_0^t E\|\nabla u^{k-1}(\tau, \psi_{t,\tau}^k(x)) - \nabla u^{k-1}(\tau, \psi_{t,\tau}^{k-1}(x))\|d\tau e^{\int_0^t \alpha(\tau)d\tau}$$

and applying the estimates from theorem 5.7 we get

$$\rho^{k+1}(t) \le e^{\int_0^t \alpha(\tau)d\tau}[L_0^1 \int_0^t \sup_x E\|u^k(\tau, \psi_{t,\tau}^k(x)) - u^{k-1}(\tau, \psi_{t,\tau}^{k-1}(x))\|d\tau$$

$$+ \int_0^t \rho^k(\tau)d\tau + \sup_x \int_0^t E\|\nabla u^{k-1}(\tau, \psi_{t,\tau}^k(x)) - \nabla u^{k-1}(\tau, \psi_{t,\tau}^{k-1}(x))\|d\tau]$$

$$+ \int_0^t \frac{1}{\sigma\sqrt{t-\tau}} C[\|\nabla u^k(\tau)\nabla u^{k-1}(\tau)\|_q + \|\nabla u^k(\tau)\nabla u^{k-1}(\tau)\|_m]$$

$$(E\|\eta^k(\tau) - \eta^{k-1}(\tau)\|_\infty^2)^{\frac{1}{2}}d\tau +$$

$$\int_0^t \frac{e^{\int_\tau^t \alpha(\theta)d\theta}}{\sigma\sqrt{t-\tau}} \sup_x E\|\nabla p^{k+1}(\tau, \psi_{t,\tau}^k(x)) - \nabla p^k(\tau, \psi_{t,\tau}^{k-1}(x))\|^2)^{\frac{1}{2}}d\tau.$$

To derive the estimate for the last term we recall (see lemma 5.1) that for $1 < q < \frac{3}{2}$ the inequality

$$\|\nabla p^k(t,x) - \nabla p^k(t,y)\| \le \|\nabla^2 p^k(t)\|_\infty\|x - y\| \le$$

$$C[\|\gamma^k(t)\|_{q,loc} + [\gamma^k(t)]_{\alpha,G}]\|x - y\|$$

holds and as a result we obtain

$$E\|\nabla p^k(\tau, \psi_{t,\tau}^k(x)) - \nabla p^k(\tau, \psi_{t,\tau}^{k-1}(x))\| \le$$

$$C[\beta(\tau) + \Theta(\tau)]E\|\psi_{t,\tau}^k(x) - \psi_{t,\tau}^{k-1}(x)\|.$$

In addition

$$\|\nabla p^{k+1}(t) - \nabla p^k(t)\|_\infty \le C_{qm}[\|\gamma^{k+1}(t) - \gamma^k(t)\|_{q,loc}+$$

$$\|\gamma^{k+1}(t) - \gamma^k(t)\|_{m,loc}] \le C_{qm}\alpha(t)[\|\nabla u^{k+1}(t) - \nabla u^k(t)\|_{q,loc}+$$

$$\|\nabla u^k(t) - \nabla u^{k-1}(t)\|_{q,loc}+$$

$$\|\nabla u^{k+1}(t) - \nabla u^k(t)\|_{m,loc} + \|\nabla u^k(t) - \nabla u^{k-1}(t)\|_{m,loc}].$$

It results from (5.50) that

$$n^{k+1}(t,x) \le C(t)[\int_0^t E\|\nabla u^k(\tau,\psi_{t,\tau}^k(x)) - \nabla u^{k-1}(\tau,\psi_{t,\tau}^{k-1}(x))\|d\tau+$$

$$\int_0^t n^k(\tau,x)d\tau + \int_0^t \frac{1}{\sigma\sqrt{t-\tau}}C_1[\|\nabla u^k(\tau)\nabla u^{k-1}(\tau)\|_q+$$

$$\|\nabla u^k(\tau)\nabla u^{k-1}(\tau)\|_m]^r (E\|\eta^{x,k}(\tau) - \eta^{x,k-1}(\tau)\|^2)^{\frac{1}{2}}d\tau$$

$$+ \int_0^t \frac{1}{\sigma\sqrt{t-\tau}}e^{\int_\tau^t \alpha(\theta)d\theta}(E\|\nabla p^{k+1}(\tau,\psi_{t,\tau}^k(x)) - \nabla p^k(\tau,\psi_{t,\tau}^{k-1}(x))\|^2)^{\frac{1}{2}}d\tau.$$

Note that by the Hölder inequality we can prove that for any positive $f(\tau) \in L^r$ and $\frac{1}{m_1} + \frac{1}{r} = 1$

$$\int_G [\int_0^t f(\tau,x)d\tau]^r dx \le \int_G t^{\frac{r}{m_1}} \int_0^t f^r(\tau,x)d\tau dx =$$

$$t^{\frac{r}{m_1}} \int_0^t \int_G f^r(\tau,x)dx d\tau$$

and for $\frac{1}{m_1} + \frac{1}{r} = 1$ and $m_1 < 2$ we have

$$\int_G [\int_0^t \frac{1}{\sigma\sqrt{t-\tau}}f(\tau,x)d\tau]^r dx \le t^{\frac{r(2-m_1)}{2m_1}} \int_0^t \int_G f^r(\tau,x)dx d\tau. \quad (5.51)$$

Then from (5.50) and (5.51) we have for $r > 2$

$$\zeta_r^{k+1}(t) \le C_2(t)[\int_0^t \int_G [E\|u^k(\tau,\psi_{t,\tau}^k(x)) - u^{k-1}(\tau,\psi_{t,\tau}^{k-1}(x))\|^r dx d\tau]+$$

$$\int_0^t \zeta_r^k(\tau)d\tau + \int_0^t \int_G \|\nabla u^{k-1}(\tau,\psi_{t,\tau}^k(x)) - \nabla u^{k-1}(\tau,\psi_{t,\tau}^{k-1}(x))\|^r dx d\tau]$$

$$+ \int_0^t \frac{1}{\sigma\sqrt{t-\tau}}C[[\|\nabla u^k(\tau)\nabla u^{k-1}(\tau)\|_q + \|\nabla u^k(\tau)\nabla u^{k-1}(\tau)\|_m]^r$$

$$\int_G (E\|\eta^{x,k}(\tau) - \eta^{x,k-1}(\tau)\|^2)^{\frac{r}{2}}dx]d\tau + \int_0^t \frac{1}{\sigma\sqrt{t-\tau}}e^{\int_\tau^t \alpha(\theta)d\theta}$$

$$\int_G (E\|\nabla p^{k+1}(\tau,\psi_{t,\tau}^k(x)) - \nabla p^k(\tau,\psi_{t,\tau}^{k-1}(x))\|^2)^{\frac{r}{2}}dx d\tau.$$

In addition for $m_r^k(t) = \|u^k(t) - u^{k-1}(t)\|_{r,loc}$ using the apriori estimates proved in lemmas 5.2 – 5.8 and theorem 5.9 we obtain

$$m_r^{k+1}(t) \le C(t)[(\int_0^t \int_G E\|u^k(\tau,\psi_{t,\tau}^k(x)) - u^{k-1}(\tau,\psi_{t,\tau}^{k-1}(x))\|^r dx d\tau)^{\frac{1}{r}}$$

$$+(\frac{1}{\sigma}t^{\frac{1}{m_1}-\frac{1}{2}} \int_0^t \int_G E\|\nabla u^{k+1}(\tau,\psi_{t,\tau}^{k+1}(x))\nabla u^k(\tau,\psi_{t,\tau}^k(x))-$$

$$\nabla u^k(\tau,\psi_{t,\tau}^k(x))\nabla u^{k-1}(\tau,\psi_{t,\tau}^{k-1}(x))\|^r dx d\tau)^{\frac{1}{r}}] \le$$

$$C_1(t)[(\int_0^t m_r^k(\tau)d\tau)^{\frac{1}{r}} + (\int_0^t \int_G \alpha(\tau)E\|\psi_{t,\tau}^k(x) - \psi_{t,\tau}^{k-1}(x)\|^r dx d\tau)^{\frac{1}{r}} +$$

$$\frac{1}{\sigma} t^{\frac{1}{m_1} - \frac{1}{2}} (\int_0^t [\rho^{k+1}(\tau) + \rho^k(\tau)]\zeta_r^k(\tau)d\tau)^{\frac{1}{r}}].$$

Since u^k and ∇u^k are uniformly bounded on $[0, T_1]$ and

$$\|\nabla u^1(t, \cdot) - \nabla u_0(\cdot)\|_{r,loc} \le const < \infty,$$

both for $r = m$ and $r = q$ we obtain that there exists a bounded on $[0, T_1]$ positive function $C_2(t)$ such that the function $\kappa^n(t) = \rho^n(t) + \zeta_m^n(t) + m_r^n$ satisfies the estimate

$$\kappa^n(t) \le \frac{[C_2(t)]^n}{n!}$$

and hence $\lim_{n \to \infty} \kappa^n(t) = 0$, since $C_2(t)$ is bounded on $[0, T_1]$. Finally we obtain that for each $t \in [0, T_1)$ the family $u^n(t, \cdot)$ uniformly converges to a limiting function $u(t, \cdot) \in C^{1,\alpha} \cap L_{m,loc}$. In addition, we can check that the limiting function $\nabla u(t, x)$ is Lipschitz continuous in x. In fact, by lemma 2.8 and theorem 2.9 for each $t \in [0, T_1]$, we have for any $x, y \in G$

$$\|\nabla u^n(t, x) - \nabla u^n(t, y)\| \le N(t)\|x - y\|,$$

where $N(t)$ and T_1 were defined above in lemmas 5.8 and theorem 5.7 respectively and the estimate is uniform in n. This allows to state that the limiting function is Lipschitz continuous as well.

To prove the uniqueness of the solution of (2.8)-(2.10) constructed above we assume first that there exist two solutions $u_1(t, x)$, $u_2(t, x)$ to (5.7)-(5.9) possessing the same initial data $u_1(0, x) = u_2(0, x) = u_0(x)$.

Computations similar to those used to prove the convergence of the family $(u^n(t), \nabla u^n(t))$ allow to check that

$$[\nabla u_1(t) - \nabla u_2(t)]_{\alpha,G} = 0 \quad \text{and} \quad \|\nabla u_1(t) - \nabla u_2(t)\|_{m,loc} = 0.$$

Finally, we know that the Cauchy problemfor a stochastic equation with Lipschitz coefficients has a unique solution. This implies the uniqueness of the solution to (5.7)-(5.9).

Summarizing the above results we see that the following statement is valid.

Theorem 5.10. *Assume that* **C 5.2** *holds. Then there exists a unique solution* $\psi_{t,x}(s), u(t, x), p(t, x)$ *of the system (5.7)-(5.9), for all t from the interval the $[0, T_1]$, with T_1 given by theorem 5.7 and $x \in G$ for any compact $G \subset R^3$. In addition $\psi_{t,x}(s)$ is a Markov process in R^3 and $u \in C([0, T_1], C^{1,\alpha}(G)) \cap C([0, T_1], L_{q,loc} \cap L_{m,loc})$ for $1 < q < \frac{3}{2} < 3 < m < \infty$.*

To fulfill our program we have to check that the conditions of theorem 2.8 are sufficient to verify that the functions $u(t, x), p(t, x)$ given by (5.8), (5.9) define a weak solution of the Navier -Stokes system.

To this end we have to apply the results of the Kunita theory of stochastic flows.

Namely we check that given a distribution valued process $\lambda(t)$ of the form

$$\lambda(t) = u_0 - \int_0^t \nabla p^u(\tau) \circ \phi_{0,\tau}^u d\tau \qquad (5.52)$$

the function

$$\lambda(t) \circ \psi_{t,0}^u = u_0 \circ \psi_{t,0}^u - \int_0^t \nabla p^u(\tau) \circ \psi_{t,\tau}^u d\tau$$

gives rise to a solution of (5.1).

To this end we apply the generalized Ito formula [19], [20] to derive

$$\lambda(t) \circ \psi_{t,0}^u = u_0 + \int_0^t \frac{\sigma^2}{2} \Delta[u(\theta) \circ \psi_{\theta,0}^u] d\theta + \qquad (5.53)$$

$$\int_0^t \nabla[u(\theta) \circ \psi_{\theta,0}^u]\sigma dw(\theta) - \int_0^t \nabla[u(\theta) \circ \psi_{\theta,0}^u]u(\theta)d\theta - \int_0^t \nabla p^u(\theta)d\theta.$$

Note that for $Lu = -(u, \nabla)u + \frac{\sigma^2}{2}\Delta u$ we have

$$E\left[\int_{R^3} \int_0^t (L(u(\tau) \circ \psi_{\tau,s}^u(x))d\tau, h(x))dx\right] =$$

$$E\left[\int_0^t \langle u(\tau) \circ \psi_{\tau,0}^u, L^*h \rangle d\tau\right] = \int_0^t L\langle E[u(\tau \circ \psi_{\tau,0}^u)], h \rangle d\tau.$$

Hence

$$u(t) = E[\lambda(t) \circ \psi_{t,0}^u] = u_0 + \int_0^t LE[u(\tau) \circ \psi_{\tau,0}^u]d\tau - \int_0^t \nabla p^u(\tau)d\tau.$$

Differentiating each term with respect to t we can check that the function

$$u(t) = E[\lambda(t) \circ \psi_{t,0}^u] \qquad (5.54)$$

solves the Cauchy problem (5.1),(5.2).

To summarize the obtained results we can state the following assertion.

Theorem 5.11. *Assume that* **C 5.2** *holds. Then the functions* $u(t,x)$, $p(t,x)$ *given by (5.8),(5.9) are defined on the interval* $[0,T_1]$ *with* T_1 *determined by theorem 5.8 and satisfy (5.1)-(5.2) in a weak sense on this interval.*

Remark 5.12. We have proved that under condition **C 5.2** the system (5.7)-(5.9) gives rise to a weak solution of (5.1)-(5.2) . Note that if the initial data are smoother, say $u_0 \in C^{2+\alpha}$, $\alpha \in [0,1]$ similar considerations can be applied to verify that the pair $u(t,x), p(t,x)$ given by (5.8)-(5.9) stands for a classical C^2-smooth solution of (5.1)-(5.2). In fact in this case applying the generalized Ito formula for the verification assertion we may treat the action of the operator L in the classical sense rather then in the weak sense.

6 Lagrangian and stochastic approach to the Euler and the N-S system

The probabilistic approach developed in the previous section is in a sense an analogue of the Lagrangian approach to the Euler and the Navier-Stokes systems. A rather close model was constructed in papers by Constantin and Iyer [11],[12]. To make it easier to compare we rewrite the results from these papers in terms similar to those used in the previous section. We consider first the Euler system

$$\frac{\partial u}{\partial t} + (u, \nabla)u = -\nabla p, \quad u(0, x) = u_0(x), \quad x \in R^3 \qquad (6.1)$$

$$div\, u = 0. \qquad (6.2)$$

and recall that the corresponding Lagrangian path starting at y is governed by the Newton equation

$$\frac{\partial^2 \tilde{\phi}_{0,t}(y)}{\partial t^2} = F_{\tilde{\phi}}(t, y). \qquad (6.3)$$

The incompressibility condition for the map ϕ yields

$$\det(\nabla \tilde{\phi}_{0,t}(y)) = 1. \qquad (6.4)$$

The force F in (3.3) has the form

$$F_{\tilde{\phi}}(t, y) = -\nabla p(t, \tilde{\phi}_{0,t}(y)) = -[(\nabla \tilde{\phi}_{0,t}(y))^*]^{-1} \nabla[p(t, \tilde{\phi}_{0,t}(y))]. \qquad (6.5)$$

One can deduce from (6.3) that

$$\frac{\partial}{\partial t}\left[\frac{\partial \tilde{\phi}_{0,t}^k(y)}{\partial t} \frac{\partial \tilde{\phi}_{0,t}^k(y)}{\partial y_i}\right] = -\frac{\partial q(t, \tilde{\phi}_{0,t}(y))}{\partial y_i}, \qquad (6.6)$$

where

$$q(t, y) = p(t, y) - \frac{1}{2}\left\|\frac{\partial \tilde{\phi}_{0,t}(y)}{\partial t}\right\|^2. \qquad (6.7)$$

We recall that in (6.6) and below summation over the repeated indices is assumed. Integrating (6.6) in time we get

$$\frac{\partial \tilde{\phi}_{0,t}^k(y)}{\partial t} \frac{\partial \tilde{\phi}_{0,t}^k(y)}{\partial y_i} = u_0(y) - \frac{\partial n(t, \tilde{\phi}_{0,t}(y))}{\partial y_i}, \qquad (6.8)$$

where

$$n(t, y) = \int_0^t q(\tau, y)d\tau \qquad (6.9)$$

and

$$u_0(y) = \frac{\partial \tilde{\phi}_{0,t}(y)}{\partial t}\Big|_{t=0} \qquad (6.10)$$

is the initial velocity.

Consider the inverse diffeomorphism $\tilde{\psi}_{t,0} = [\tilde{\phi}_{0,t}]^{-1}$, come back to (6.7), multiply it by $[\nabla \tilde{\psi}_{t,0}]$ and put $y = \tilde{\psi}_{t,0}(x)$. As a result we obtain by the chain rule the relation

$$u^i(t, x) = (u_0^j(\tilde{\psi}_{t,0}(x))\nabla_{x_i}\tilde{\psi}_{t,0}^j(x) - \int_0^t \nabla_{x_i}q(\tau, \tilde{\psi}_{t,\tau}(x))d\tau. \qquad (6.11)$$

The equation (6.11) shows that the general Euler velocity may be written in the form that generalizes the Clebsch variable representation

$$u = [\nabla \tilde{\psi}_{t,0}]^* C - \nabla n,$$

where $C = u_0(\psi_{t,0}(x))$ is an active vector and n is defined by the incompressibility condition $div u = 0$.

Note that a vector A is called active if

$$\frac{d}{dt} A = \frac{\partial A}{\partial t} + (u, \nabla)A = 0.$$

It is easy to check by the chain rule that

$$\frac{d}{dt}\tilde{\psi}_{t,\theta}(x) = \frac{\partial \tilde{\psi}_{t,\theta}(x)}{\partial t} + (u, \nabla)\tilde{\psi}_{t,\theta}(x) = 0, \qquad (6.12)$$

that is $\psi_{t,0}(x)$ is an active vector.

Hence the Euler equations are equivalent to the system consisting of (3.9) and the following relation

$$\Delta n(t, x) = \frac{\partial}{\partial x_i}\{u_0^k(\tilde{\psi}_{t,0}(x))\frac{\partial \tilde{\psi}_{t,0}^k(x)}{\partial x_i}\}, \qquad (6.13)$$

where n is given by (6.9).

Now one can assume the periodic boundary conditions or the zero boundary conditions at infinity. Note that in the periodic case $n(t, x), u(t, x)$ and

$$\delta(t, x) = x - \tilde{\psi}_{t,0}(x) \qquad (6.14)$$

are periodic functions in each spatial direction. Finally due to $div u = 0$ one can rewrite the equation of state (6.11) in the form

$$u(t) = \Pi\{u_0^j(\tilde{\psi}_{t,0})\nabla \tilde{\psi}_{t,0}^j\} = \Pi\{[\nabla \tilde{\psi}_{t,0}]^* u_0(\tilde{\psi}_{t,0})\}, \qquad (6.15)$$

where $\Pi = I - \nabla \Delta^{-1} \nabla$ is the Leray-Hodge projector (with corresponding boundary conditions) on divergence free vector fields. The Euler pressure is determined up to additive constants by

$$p(t, x) = \frac{\partial n(t, x)}{\partial t} + (u(t, x), \nabla)n(t, x) + \frac{1}{2}\|u(t, x)\|^2.$$

Note that (6.11), (6.12) made a closed system and may be used to determine $u(t)$.

Let us compare (6.11), (6.12) with the alternative representation for the state $u(t)$ developed in the previous section.

To this end we choose $\phi_{0,t} : y \to \phi_{0,t}(y)$ to be a volume preserving diffeomorphism that satisfies the equation

$$d\phi_{0,\tau}(y) = u(t - \tau, \phi_{0,\tau}(y))d\tau, \quad \phi_{0,0}(y) = y, \qquad (6.16)$$

with div $u(t) = 0$.

Consider the system

$$d\psi_{t,\theta}(x) = -u(\theta, \psi_{t,\theta}(x))d\theta, \quad \psi_{t,t}(x) = x, \qquad (6.17)$$

$$u(t,x) = u_0(\psi_{t,0}(x)) - \int_0^t \nabla p(\tau, \psi_{t,\tau}(x))d\tau, \qquad (6.18)$$

$$-2p(t,x) = E[\int_0^\infty \gamma(t, x + B(\tau))d\tau], \qquad (6.19)$$

where γ is given by (1.3).

If the fields $u(t,x), p(t,x)$ are regular enough then we may construct the representation of the solution to the Euler system in the form (6.18), (6.19).

To check this we consider a volume preserving diffeomorphism $\phi_{0,t}$: $y \to \tilde{\phi}_{0,t}(y)$ that satisfies (6.16).

Next we consider the system

$$d\psi_{t,\theta}(x) = -u(\theta, \psi_{t,\theta}(x))d\theta, \quad \psi_{t,t}(x) = x, \qquad (6.20)$$

$$u(t,x) = u_0(\psi_{t,0}(x)) - \int_0^t \nabla p(\tau, \psi_{t,\tau}(x))d\tau, \qquad (6.21)$$

$$-2p(t,x) = E[\int_0^\infty \gamma(t, x + B(\tau))d\tau], \qquad (6.22)$$

where γ is given by (1.4).

If $u(t,x)$ is regular enough then we may construct the representation of the solution to the Euler system (6.1), (6.2) in the form (6.21), (6.22).

To this end we consider a vector field $\lambda(t)$ satisfying the equation

$$\frac{d\lambda(t)}{dt} = -\nabla p(t) \circ \phi_{0,t} \quad \lambda(0) = u_0,$$

where $\phi_{0,t}$ satisfies the ODE (6.16), and let the process $\psi_{t,0}$ be its inverse. Applying the Kunita approach [19] to the process $\psi_{t,0}$ we can verify that $\psi_{t,0}$ along with (6.20) satisfies the equation

$$\psi_{t,\tau}(x) = x + \int_\tau^t \nabla \phi_{\theta,t}^g(\psi_{t,\theta})^{-1} u(\theta, x)d\theta, \qquad (6.23)$$

that allows to prove that $u(t)$ given by (6.21) satisfies (6.1).

Comparing (6.15) and (6.21) we note that they give different expressions for the velocity field. Actually (6.21) includes the Euler pressure $p(t,x)$ instead of $q(t,x)$ used in (6.15). Besides the probabilistic representation of the solution p to the Poisson equation

$$-\Delta p = \nabla_i u_k \nabla_k u_i$$

is used instead of the Leray projection.

Coming back to the Navier-Stokes system ((5.1),(5.2) we recall here the approach due to Constantin and Iyer [12]. The stochastic counterpart of the Navier-Stokes equations in the version of Iyer [25] looks like the following.

Consider the closed stochastic system

$$d\phi_{0,\theta} = u(\theta, \phi_{0,\theta})dt + \sigma dw(\theta), \quad \phi_{0,0}(y) = y, \qquad (6.24)$$

$$\psi_{\theta,0} = [\phi_{0,\theta}]^{-1}, \qquad (6.25)$$

$$u(t) = E\Pi[(\nabla\psi_{t,0})(u_0 \circ \psi_{t,0})]. \tag{6.26}$$

The existence and uniqueness of the solution to this system is proved in [12] by the successive approximation technique. As a result the authors constructed a strong local in time solution of the Cauchy problem for the Navier-Stokes system for regular enough initial data.

The main result due to Constantin and Iyer reads as follows

Theorem 6.1. *Let $k \geq 1$ and $u_0 \in C^{k+1,\alpha}$ be divergence free. Then there exists a time interval $[0, T]$ with $T = T(k, \alpha, L, \|u_0\|_{k+1,\alpha})$ but independent of viscosity σ and a pair $\phi_{0,t}(x), u(t, x)$ such that $u \in C([0, T], C^{k+1,\alpha})$ and (u, ϕ) satisfy (6.24)-(6.26). Further there exists $U = U(k, \alpha, L, \|u_0\|_{k+1,\alpha})$ such that $\|u(t)\|_{k+1,\alpha} \leq U$ for $t \in [0, T]$ and u satisfies the N-S system.*

As we have mentioned above an approach close to the one of [12] was developed in our previous paper [7]. Both these approaches allow to construct a classical (local in time) solution to the Cauchy problem for the Navier-Stokes system and prove the uniqueness of the solution.

On the other hand the approach developed in section 5 allows to construct a weak (local in time) solution to (5.1), (5.2) and prove the uniqueness of this solution in the corresponding functional classes.

The stochastic counterpart of the Navier-Stokes system considered in section 2 has the form

$$d\psi_{t,\theta}(x) = -u(\theta, \psi_{t,\theta}(x)d\theta + \sigma d\hat{w}(\theta), \quad \psi_{t,t}(x) = x, \tag{6.27}$$

$$u(t, x) = E[u_0(\psi_{t,0}(x)) - \int_0^t \nabla p(\tau, \psi_{t,\tau}(x))d\tau], \tag{6.28}$$

$$2p(t, x) = -\int_0^\infty E[\gamma(t, x + B(\tau))]d\tau. \tag{6.29}$$

Note that we can use the relation

$$-2\nabla p(t, x) = E[\int_0^\infty \frac{1}{\tau}\gamma(t, x + B(\tau))B(\tau)d\tau] \tag{6.30}$$

to eliminate the pressure from the above system (6.27) – (6.29).

We can see that the difference between (6.27) – (6.29) and (6.24) – (6.26) has the same nature as the difference between (6.11), (6.12) and (6.17) – (6.19).

Finally we note that the approach developed in section 5 allows us to construct both strong (classical) and weak (distributional) solutions of the Cauchy problem for the N-S system.

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